CHAPTER 12

PLANE WAVES AT BOUNDARIES AND IN DISPERSIVE MEDIA

In Chapter 11, we considered basic electromagnetic wave principles. We learned how to mathematically represent waves as functions of frequency, medium properties, and electric field orientation. We also learned how to calculate the wave velocity, attenuation, and power. In this chapter, we consider wave reflection and transmission at planar boundaries between media having different properties. Our study will allow any orientation between the wave and boundary, and will also include the important cases of multiple boundaries. We will also study the practical case of waves that carry power over a finite band of frequencies, as would occur, for example, in a modulated carrier. We will consider such waves in *dispersive* media, in which some parameter that affects propagation (permittivity for example) varies with frequency. The effect of a dispersive medium on a signal is of great importance, since the signal envelope will change its shape as it propagates. This can occur to an extent that at the receiving end, detection and faithful representation of the original signal become problematic. Dispersion thus becomes an important limiting factor in allowable propagation distances, in a similar way that we found to be true for attenuation.

12.1 REFLECTION OF UNIFORM PLANE WAVES AT NORMAL INCIDENCE

In this section we will consider the phenomenon of reflection which occurs when a uniform plane wave is incident on the boundary between regions composed of two different materials. The treatment is specialized to the case of *normal incidence*, in which the wave propagation direction is perpendicular to the boundary. In later sections we remove this restriction. We shall establish expressions for the wave that is reflected from the interface and for that which is transmitted from one region into the other. These results will be directly applicable to impedance-matching problems in ordinary transmission lines as well as to waveguides and other more exotic transmission systems.

We again assume that we have only a single vector component of the electric field intensity. Referring to Fig. 12.1, we define region 1 (ϵ_1, μ_1) as the half-space for which $z < 0$; region 2 (ϵ_2, μ_2) is the half-space for which $z > 0$. Initially we establish the wave traveling in the $+z$ direction in region 1,

$$E_{x1}^{+}(z,t) = E_{x10}^{+}\, e^{-\alpha_1 z}\cos(\omega t - \beta_1 z)$$

or

$$E_{xs1}^{+} = E_{x10}^{+}\, e^{-jk_1 z} \tag{1}$$

where we take E_{x10}^{+} as real. The subscript 1 identifies the region and the superscript $+$ indicates a positively traveling wave. Associated with $E_{x1}^{+}(z,t)$ is a magnetic field

$$H_{ys1}^{+} = \frac{1}{\eta_1}\, E_{x10}^{+}\, e^{-jk_1 z} \tag{2}$$

where k_1 and η_1 are complex unless ϵ_1'' (or σ_1) is zero. This uniform plane wave in region 1 which is traveling toward the boundary surface at $z = 0$ is called the *incident* wave. Since the direction of propagation of the incident wave is perpendicular to the boundary plane, we describe it as normal incidence.

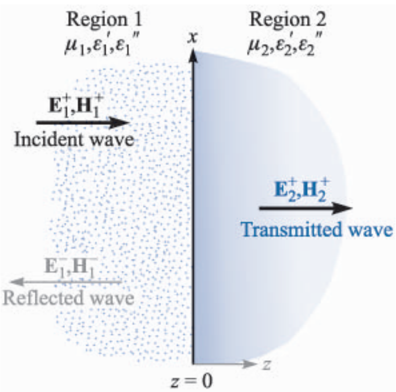

FIGURE 12.1
A wave $\mathbf{E}_1^+$ incident on a plane boundary establishes a reflected wave $\mathbf{E}_1^-$ and a transmitted wave $\mathbf{E}_2^+$.

We now recognize that energy may be transmitted across the boundary surface at $z = 0$ into region 2 by providing a wave moving in the $+z$ direction in that medium,

$$E^+_{xs2} = E^+_{x20}\, e^{-jk_2 z} \tag{3}$$

$$H^+_{ys2} = \frac{1}{\eta_2} E^+_{x20}\, e^{-jk_2 z} \tag{4}$$

This wave which moves away from the boundary surface into region 2 is called the *transmitted* wave; note the use of the different propagation constant k_2 and intrinsic impedance η_2.

Now we must try to satisfy the boundary conditions at $z = 0$ with these assumed fields. E_x is a tangential field; therefore the **E** fields in regions 1 and 2 must be equal at $z = 0$. Setting $z = 0$ in (1) and (3) would require that $E^+_{x10} = E^+_{x20}$. H_y is also a tangential field, however, and must be continuous across the boundary (no current sheets are present in real media). When we let $z = 0$ in (2) and (4), however, we find that we must have $E^+_{x10}/\eta_1 = E^+_{x20}/\eta_2$; since $E^+_{x10} = E^+_{x20}$, then $\eta_1 = \eta_2$. But this is a very special condition that does not fit the facts in general, and we are therefore unable to satisfy the boundary conditions with only an incident and a transmitted wave. We require a wave traveling away from the boundary in region 1, as shown in Fig. 12.1; this is called a *reflected* wave,

$$E^-_{xs1} = E^-_{x10}\, e^{jk_1 z} \tag{5}$$

$$H^-_{ys1} = -\frac{E^-_{x10}}{\eta_1}\, e^{jk_1 z} \tag{6}$$

where E^-_{x10} may be a complex quantity. Since this field is traveling in the $-z$ direction, $E^-_{xs1} = -\eta_1 H^-_{ys1}$, for the Poynting vector shows that $\mathbf{E}^-_1 \times \mathbf{H}^-_1$ must be in the $-\mathbf{a_z}$ direction.

The boundary conditions are now easily satisfied, and in the process the amplitudes of the transmitted and reflected waves may be found in terms of E^+_{x10}. The total electric field intensity is continuous at $z = 0$,

$$E_{xs1} = E_{xs2} \qquad (z = 0)$$

or

$$E^+_{xs1} + E^-_{xs1} = E^+_{xs2} \qquad (z = 0)$$

Therefore

$$\boxed{E^+_{x10} + E^-_{x10} = E^+_{x20}} \tag{7}$$

Furthermore,

$$H_{ys1} = H_{ys2} \qquad (z = 0)$$

or

$$H^+_{ys1} + H^-_{ys1} = H^+_{ys2} \qquad (z = 0)$$

and therefore

$$\boxed{\frac{E^+_{x10}}{\eta_1} - \frac{E^-_{x10}}{\eta_1} = \frac{E^+_{x20}}{\eta_2}} \tag{8}$$

Solving (8) for E^+_{x20} and substituting into (7), we find

$$E^+_{x10} + E^-_{x10} = \frac{\eta_2}{\eta_1} E^+_{x10} - \frac{\eta_2}{\eta_1} E^-_{x10}$$

or

$$E^-_{x10} = E^+_{x10} \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

The ratio of the amplitudes of the reflected and incident electric fields is called the *reflection coefficient* and is designated by Γ (gamma),

$$\boxed{\Gamma = \frac{E^-_{x10}}{E^+_{x10}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}} \tag{9}$$

The reflection coefficient may be complex, in which case there is a phase shift in the reflected wave.

The relative amplitude of the transmitted electric field intensity is found by combining (9) and (7), to yield the *transmission coefficient*, τ:

$$\boxed{\tau = \frac{E^+_{x20}}{E^+_{x10}} = \frac{2\eta_2}{\eta_1 + \eta_2} = 1 + \Gamma} \tag{10}$$

Let us see how these results may be applied to several special cases. We first let region 1 be a perfect dielectric and region 2 be a perfect conductor. Then, since σ_2 is infinite,

$$\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon'_2}} = 0$$

and from (10),

$$E^+_{x20} = 0$$

No time-varying fields can exist in the *perfect* conductor. An alternate way of looking at this is to note that the skin depth is zero.

Since $\eta_2 = 0$, then (9) shows that

$$\Gamma = -1$$

and

$$E_{x10}^{+} = -E_{x10}^{-}$$

The incident and reflected fields are of equal amplitude, and so all the incident energy is reflected by the perfect conductor. The fact that the two fields are of opposite sign indicates that at the boundary (or at the moment of reflection) the reflected field is shifted in phase by 180° relative to the incident field. The total **E** field in region 1 is

$$\begin{aligned} E_{xs1} &= E_{xs1}^{+} + E_{xs1}^{-} \\ &= E_{x10}^{+} e^{-j\beta_1 z} - E_{x10}^{+} e^{j\beta_1 z} \end{aligned}$$

where we have let $jk_1 = 0 + j\beta_1$ in the perfect dielectric. These terms may be combined and simplified,

$$\begin{aligned} E_{xs1} &= \left(e^{-j\beta_1 z} - e^{j\beta_1 z}\right) E_{x10}^{+} \\ &= -j2 \sin(\beta_1 z)\, E_{x10}^{+} \end{aligned} \tag{11}$$

Multiplying (11) by $e^{j\omega t}$ and taking the real part, we may drop the s subscript and obtain the real instantaneous form:

$$\boxed{E_{x1}(z, t) = 2E_{x10}^{+} \sin(\beta_1 z) \sin(\omega t)} \tag{12}$$

This total field in region 1 is not a traveling wave, although it was obtained by combining two waves of equal amplitude traveling in opposite directions. Let us compare its form with that of the incident wave,

$$E_{x1}(z, t) = E_{x10}^{+} \cos(\omega t - \beta_1 z) \tag{13}$$

Here we see the term $\omega t - \beta_1 z$ or $\omega(t - z/v_{p1})$, which characterizes a wave traveling in the $+z$ direction at a velocity $v_{p1} = \omega/\beta_1$. In (12), however, the factors involving time and distance are separate trigonometric terms. At all planes for which $\beta_1 z = m\pi$, E_{x1} is zero for all time. Furthermore, whenever $\omega t = m\pi$, E_{x1} is zero everywhere. A field of the form of (12) is known as a *standing wave*.

The planes on which $E_{x1} = 0$ are located where

$$\beta_1 z = m\pi \qquad (m = 0, \pm 1, \pm 2, \ldots)$$

Thus

$$\frac{2\pi}{\lambda_1} z = m\pi$$

and

$$z = m\frac{\lambda_1}{2}$$

Thus $E_{x1} = 0$ at the boundary $z = 0$ and at every half-wavelength from the boundary in region 1, $z < 0$, as illustrated in Fig. 12.2.

Since $E^+_{xs1} = \eta_1 H^+_{ys1}$ and $E^-_{xs1} = -\eta_1 H^-_{ys1}$, the magnetic field is

$$H_{ys1} = \frac{E^+_{x10}}{\eta_1}\left(e^{-j\beta_1 z} + e^{j\beta_1 z}\right)$$

or

$$\boxed{H_{y1}(z, t) = 2\frac{E^+_{x10}}{\eta_1}\cos(\beta_1 z)\cos(\omega t)} \tag{14}$$

This is also a standing wave, but it shows a maximum amplitude at the positions where $E_{x1} = 0$. It is also 90° out of time phase with E_{x1} everywhere. Thus no average power is transmitted in either direction.

Let us now consider perfect dielectrics in both regions 1 and 2; η_1 and η_2 are both real positive quantities and $\alpha_1 = \alpha_2 = 0$. Equation (9) enables us to calcu-

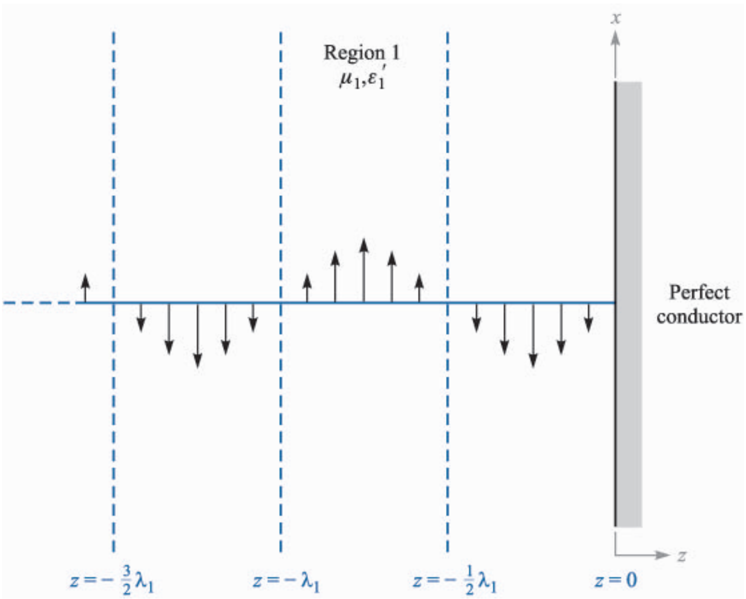

FIGURE 12.2
The instantaneous values of the total field E_{x1} are shown at $t = \pi/2$. $E_{x1} = 0$ for all time at multiples of one half-wavelength from the conducting surface.

late the reflection coefficient and find E_{x1}^- in terms of the incident field E_{x1}^+. Knowing E_{x1}^+ and E_{x1}^-, we then find H_{y1}^+ and H_{y1}^-. In region 2, E_{x2}^+ is found from (10), and this then determines H_{y2}^+.

Example 12.1

As a numerical example let us select

$$\eta_1 = 100 \quad \Omega$$
$$\eta_2 = 300 \quad \Omega$$
$$E_{x10}^+ = 100 \quad \text{V/m}$$

and calculate values for the incident, reflected, and transmitted waves.

Solution. The reflection coefficient is

$$\Gamma = \frac{300 - 100}{300 + 100} = 0.5$$

and thus

$$E_{x10}^- = 50 \quad \text{V/m}$$

The magnetic field intensities are

$$H_{y10}^+ = \frac{100}{100} = 1.00 \quad \text{A/m}$$

$$H_{y10}^- = -\frac{50}{100} = -0.50 \quad \text{A/m}$$

The incident power density is

$$\mathcal{P}_{1,av}^+ = \frac{1}{2} E_{x10}^+ H_{y10}^+ = 100 \quad \text{W/m}^2$$

while

$$\mathcal{P}_{1,av}^- = -\frac{1}{2} E_{x10}^- H_{y10}^- = 25.0 \quad \text{W/m}^2$$

In region 2, using (10)

$$E_{x20}^+ = \tau E_{x10}^+ = 150 \quad \text{V/m}$$

and

$$H_{y20}^+ = \frac{150}{300} = 0.500 \quad \text{A/m}$$

Thus

$$\mathcal{P}_{2,av}^+ = \frac{1}{2} E_{x20}^+ H_{y20}^+ = 75.0 \quad \text{W/m}^2$$

Note that energy is conserved:

$$\mathcal{P}_{1,av}^+ = \mathcal{P}_{1,av}^- + \mathcal{P}_{2,av}^+$$

We can formulate a general rule on the transfer of power through reflection and transmission by using Eq. (57) from Chapter 11:

$$\mathcal{P}_{av} = \frac{1}{2}\text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\}$$

We consider the same field vector and interface orientations as before, but we consider the general case of complex impedances. For the incident power density, we have

$$\mathcal{P}_{1,av}^{+} = \frac{1}{2}\text{Re}\left\{E_{x10}^{+}\,H_{y10}^{+*}\right\} = \frac{1}{2}\text{Re}\left\{E_{x10}^{+}\,\frac{1}{\eta_1^*}\,E_{x10}^{+*}\right\} = \frac{1}{2}\text{Re}\left\{\frac{1}{\eta_1^*}\right\}|E_{x10}^{+}|^2$$

The reflected power density is then

$$\mathcal{P}_{1,av}^{-} = -\frac{1}{2}\text{Re}\left\{E_{x10}^{-}\,H_{y10}^{-*}\right\} = \frac{1}{2}\text{Re}\left\{\Gamma E_{x10}^{+}\,\frac{1}{\eta_1^*}\,\Gamma^* E_{x10}^{+*}\right\} = \frac{1}{2}\text{Re}\left\{\frac{1}{\eta_1^*}\right\}|E_{x10}^{+}|^2\,|\Gamma|^2$$

We thus find the general relation between the reflected and incident power:

$$\boxed{\mathcal{P}_{1,av}^{-} = |\Gamma|^2\mathcal{P}_{1,av}^{+}} \tag{15}$$

In a similar way, we find the transmitted power:

$$\mathcal{P}_{2,av}^{+} = \frac{1}{2}\text{Re}\left\{E_{x20}^{+}\,H_{y20}^{+*}\right\} = \frac{1}{2}\text{Re}\left\{\tau E_{x10}^{+}\,\frac{1}{\eta_2^*}\,\tau^* E_{x10}^{+*}\right\} = \frac{1}{2}\text{Re}\left\{\frac{1}{\eta_2^*}\right\}|E_{x10}^{+}|^2\,|\tau|^2$$

and so we see that the incident and transmitted powers are related through

$$\mathcal{P}_{2,av}^{+} = \frac{\text{Re}\{1/\eta_2^*\}}{\text{Re}\{1/\eta_1^*\}}\,|\tau|^2\,\mathcal{P}_{1,av}^{+} = \left|\frac{\eta_1}{\eta_2}\right|^2\left(\frac{\eta_2+\eta_2^*}{\eta_1+\eta_1^*}\right)|\tau|^2\,\mathcal{P}_{1,av}^{+} \tag{16}$$

Eq. (16) is a relatively complicated way to calculate the transmitted power, unless the impedances are real. It is easier to take advantage of energy conservation by noting that whatever power is not reflected must be transmitted. Eq. (15) can thus be used to find

$$\boxed{\mathcal{P}_{2,av}^{+} = \left(1-|\Gamma|^2\right)\mathcal{P}_{1,av}^{+}} \tag{17}$$

As would be expected (and which must be true), Eq. (17) can also be derived from Eq. (16).

✔ **D12.1.** A 1 MHz uniform plane wave is normally incident onto a freshwater lake ($\epsilon_R' = 78$, $\epsilon_R'' = 0$, $\mu_R = 1$). Determine the fraction of the incident power that is (a) reflected and (b) transmitted; (c) determine the amplitude of the electric field that is transmitted into the lake.

Ans. 0.63; 0.37; 0.20 V/m.

12.2 STANDING WAVE RATIO

One of the measurements that is easily made on transmission systems is the *relative* amplitude of the electric or magnetic field intensity through use of a probe. A small coupling loop will give an indication of the amplitude of the magnetic field, while a slightly extended center conductor of a coaxial cable will sample the electric field. Both devices are customarily tuned to the operating frequency to provide increased sensitivity. The output of the probe is rectified and connected directly to a microammeter, or it may be delivered to an electronic voltmeter or a special amplifier. The indication is proportional to the amplitude of the sinusoidal time-varying field in which the probe is immersed.

When a uniform plane wave is traveling through a lossless region, and no reflected wave is present, the probe will indicate the same amplitude at every point. Of course, the instantaneous field which the probe samples will differ in phase by $\beta(z_2 - z_1)$ rad as the probe is moved from $z = z_1$ to $z = z_2$, but the system is insensitive to the phase of the field. The equal-amplitude voltages are characteristic of an unattenuated traveling wave.

When a wave traveling in a lossless medium is reflected by a perfect conductor, the total field is a standing wave and, as shown by Eq. (12), the voltage probe provides no output when it is located an integral number of half-wavelengths from the reflecting surface. As the probe position is changed, its output varies as $|\sin \beta z|$, where z is the distance from the conductor. This sinusoidal amplitude variation is shown in Fig. 12.3, and it characterizes a standing wave.

A more complicated situation arises when the reflected field is neither 0 nor 100 percent of the incident field. Some energy is transmitted into the second region and some is reflected. Region 1 therefore supports a field that is composed of both a traveling wave and a standing wave. It is customary to describe this field as a standing wave even though a traveling wave is also present. We shall see that the field does not have zero amplitude at any point for all time, and the degree to which the field is divided between a traveling wave and a true standing wave is expressed by the ratio of the maximum amplitude found by the probe to the minimum amplitude.

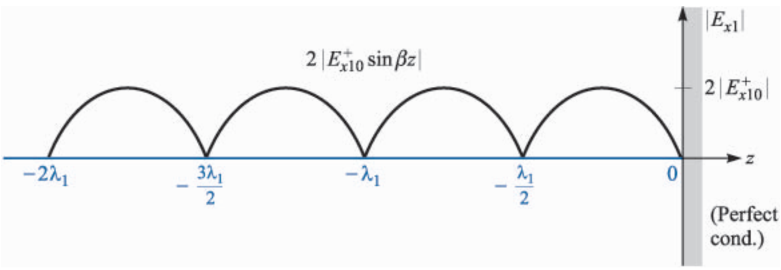

FIGURE 12.3
The standing voltage wave produced in a lossless medium by reflection from a perfect conductor varies as $|\sin \beta z|$.

Using the same fields investigated in the previous section, we combine the incident and reflected electric field intensities,

$$E_{x1} = E_{x1}^{+} + E_{x1}^{-}$$

The field E_{x1} is a sinusoidal function of t (generally with a nonzero phase angle), and it varies with z in a manner as yet unknown. We shall inspect all z to find the maximum and minimum amplitudes, and determine their ratio. We call this ratio the *standing-wave ratio*, and we shall symbolize it by s.

Let us now go through the mechanics of this procedure for the case in which medium 1 is a perfect dielectric, $\alpha_1 = 0$, but region 2 may be any material. We have

$$E_{xs1}^{+} = E_{x10}^{+}\, e^{-j\beta_1 z}$$

$$E_{xs1}^{-} = \Gamma E_{x10}^{+}\, e^{j\beta_1 z}$$

where

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

and η_1 is real and positive but η_2 may be complex. Thus Γ may be complex, and we allow for this possibility by letting

$$\Gamma = |\Gamma|\, e^{j\phi}$$

If region 2 is a perfect conductor, ϕ is equal to π; if η_2 is real and less than η_1, ϕ is also equal to π; and if η_2 is real and greater than η_1, ϕ is zero. The total field in region 1 is

$$E_{xs1} = \left(e^{-j\beta_1 z} + |\Gamma| e^{j(\beta_1 z + \phi)}\right) E_{x10}^{+} \tag{18}$$

We seek the maximum and minimum values of the magnitude of the complex quantity in the larger parentheses in (18). We certainly have a maximum when each term in the larger parentheses has the same phase angle; thus, for E_{x10}^{+} positive and real,

$$E_{xs1,\,max} = (1 + |\Gamma|) E_{x10}^{+} \tag{19}$$

and this occurs where

$$-\beta_1 z = \beta_1 z + \phi + 2m\pi \qquad (m = 0,\ \pm 1,\ \pm 2, \ldots) \tag{20}$$

Thus

$$\boxed{z_{max} = -\frac{1}{2\beta_1}(\phi + 2m\pi)} \tag{21}$$

Note that a field maximum is located at the boundary plane ($z = 0$) if $\phi = 0$; moreover, $\phi = 0$ when Γ is real and positive. This occurs for real η_1 and η_2 when

$\eta_2 > \eta_1$. Thus there is a voltage maximum at the boundary surface when the intrinsic impedance of region 2 is greater than that of region 1 and both impedances are real. With $\phi = 0$, maxima also occur at $z_{max} = -m\pi/\beta_1 = -m\lambda_1/2$.

For the perfect conductor $\phi = \pi$, and these maxima are found at $z_{max} = -\pi/(2\beta_1)$, $-3\pi/(2\beta_1)$, or $z_{max} = -\lambda_1/4$, $-3\lambda_1/4$, and so forth.

The minima must occur where the phase angles of the two terms in the larger parentheses in (18) differ by 180°; thus

$$E_{xs1,\,min} = (1 - |\Gamma|)E_{x10}^{+} \tag{22}$$

and this occurs where

$$-\beta_1 z = \beta_1 z + \phi + \pi + 2m\pi \qquad (m = 0,\ \pm 1,\ \pm 2, \ldots) \tag{23}$$

or

$$\boxed{z_{min} = -\frac{1}{2\beta_1}(\phi + (2m+1)\pi)} \tag{24}$$

The minima are separated by multiples of one half-wavelength (as are the maxima), and for the perfect conductor the first minimum occurs when $-\beta_1 z = 0$, or at the conducting surface. In general, an electric field minimum is found at $z = 0$ whenever $\phi = \pi$; this occurs if $\eta_2 < \eta_1$ and both are real. The general results are illustrated in Fig. 12.4.

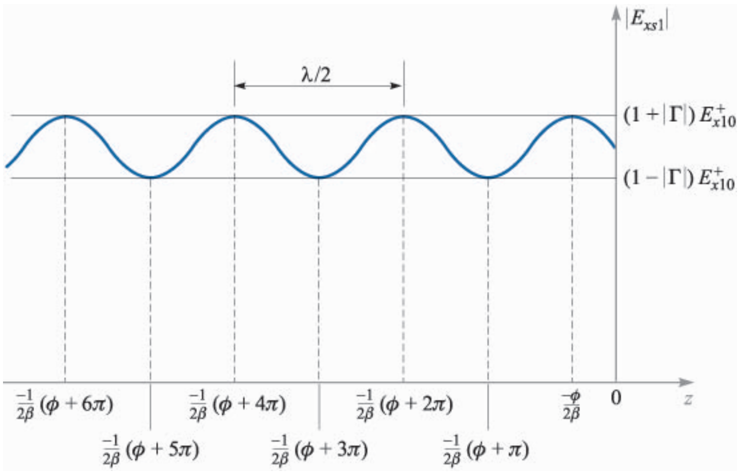

FIGURE 12.4
Plot of the magnitude of E_{xs1} as found from Eq. (18) as a function of position, z, in front of the interface (at $z = 0$). The reflection coefficient phase is ϕ, which leads to the indicated locations of maximum and minimum **E**, as found through Eqs. (21) and (24).

Further insights can be obtained by revisiting Eq. (18), which can be rewritten in the form

$$E_{xs1} = E_{x10}^{+}\left(e^{-j\phi/2}e^{-j\beta_1 z} + |\Gamma|e^{j\phi/2}e^{j\beta_1 z}\right)e^{j\phi/2}$$

As a trick, we can add and subtract $\left(|\Gamma|E_{x10}^{+}\, e^{-j\phi/2}e^{-j\beta_1 z}\right)$ to obtain

$$E_{xs1} = E_{x10}^{+}(1-|\Gamma|)e^{-j\beta_1 z} + E_{x10}^{+}|\Gamma|\left(e^{-j\phi/2}e^{-j\beta_1 z} + e^{j\phi/2}e^{j\beta_1 z}\right)e^{j\phi/2}$$

which reduces to

$$E_{xs1} = (1-|\Gamma|)E_{x10}^{+}e^{-j\beta_1 z} + 2|\Gamma|E_{x10}^{+}\, e^{j\phi/2}\cos(\beta_1 z + \phi/2) \tag{25}$$

Finally, the real instantaneous form of (25) is obtained through $E_{x1}(z,t) = \mathrm{Re}\left\{E_{xs1}e^{j\omega t}\right\}$. We find

$$\boxed{\begin{aligned} E_{x1}(z,t) = {} & (1-|\Gamma|)E_{x10}^{+}\cos(\omega t - \beta_1 z) \\ & + 2|\Gamma|E_{x10}^{+}\cos(\beta_1 z + \phi/2)\cos(\omega t + \phi/2) \end{aligned}} \tag{26}$$

Equation (26) is recognized as the sum of a traveling wave of amplitude $(1-|\Gamma|)E_{x10}^{+}$ and a standing wave having amplitude $2|\Gamma|E_{x10}^{+}$. We can visualize events as follows: The portion of the incident wave that reflects and back-propagates in region 1 interferes with an equivalent portion of the incident wave to form a standing wave. The rest of the incident wave (that does not interfere) is the traveling wave part of (26). The maximum amplitude observed in region 1 is found where the amplitudes of the two terms in (26) add directly to give $(1+|\Gamma|)E_{x10}^{+}$. The minimum amplitude is found where the standing wave achieves a null, leaving only the traveling wave amplitude of $(1-|\Gamma|)E_{x10}^{+}$. The fact that the two terms in (26) combine in this way with the proper phasing is not readily apparent, but can be confirmed by substituting z_{max} and z_{min}, as given by (21) and (24).

Example 12.2

To illustrate some of these results, let us consider a 100-V/m, 3-GHz wave that is propagating in a material having $\epsilon'_{R1} = 4$, $\mu_{R1} = 1$, and $\epsilon''_R = 0$. The wave is normally incident on another perfect dielectric in region 2, $z > 0$, where $\epsilon'_{R2} = 9$ and $\mu_{R2} = 1$ (Fig. 12.5). We seek the locations of the maxima and minima of **E**.

Solution. We calculate $\omega = 6\pi \times 10^9$ rad/s, $\beta_1 = \omega\sqrt{\mu_1\epsilon_1} = 40\pi$ rad/m, and $\beta_2 = \omega\sqrt{\mu_2\epsilon_2} = 60\pi$ rad/m. Although the wavelength would be 10 cm in air, we find here that $\lambda_1 = 2\pi/\beta_1 = 5$ cm, $\lambda_2 = 2\pi/\beta_2 = 3.33$ cm, $\eta_1 = 60\pi\,\Omega$, $\eta_2 = 40\pi\,\Omega$, and $\Gamma = (\eta_2 - \eta_1)/(\eta_2 + \eta_1) = -0.2$. Since Γ is real and negative ($\eta_2 < \eta_1$), there will be a minimum of the electric field at the boundary, and it will be repeated at half-wavelength (2.5 cm) intervals in dielectric 1. From (22), we see that $E_{xs1,\,min} = 80$ V/m.

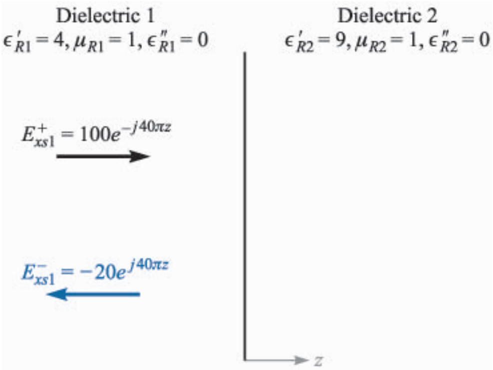

FIGURE 12.5
An incident wave, $E^+_{xs1} = 100e^{-j40\pi z}$ V/m is reflected with a reflection coefficient $\Gamma = -0.2$. Dielectric 2 is infinitely thick.

Maxima of **E** are found at distances of 1.25, 3.75, 6.25, ... cm from $z = 0$. These maxima all have amplitudes of 120 V/m, as predicted by (19).

There are no maxima or minima in region 2 since there is no reflected wave there.

The ratio of the maximum to minimum amplitudes is called the *standing wave ratio*:

$$\boxed{s = \frac{E_{xs1,\,max}}{E_{xs1,\,min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}} \tag{27}$$

Since $|\Gamma| < 1$, s is always positive and greater than or equal to unity. For the example above,

$$s = \frac{1 + |-0.2|}{1 - |-0.2|} = \frac{1.2}{0.8} = 1.5$$

If $|\Gamma| = 1$, the reflected and incident amplitudes are equal, all the incident energy is reflected, and s is infinite. Planes separated by multiples of $\lambda_1/2$ can be found on which E_{x1} is zero at all times. Midway between these planes, E_{x1} has a maximum amplitude twice that of the incident wave.

If $\eta_2 = \eta_1$, then $\Gamma = 0$, no energy is reflected, and $s = 1$; the maximum and minimum amplitudes are equal.

If one-half the incident power is reflected, $|\Gamma|^2 = 0.5$, $|\Gamma| = 0.707$, and $s = 5.83$.

✔ **D12.2.** What value of s results when $\Gamma = \pm 1/2$?

Ans. 3

Since the standing-wave ratio is a ratio of amplitudes, the relative amplitudes provided by a probe permit its use to determine s experimentally.

Example 12.3

A uniform plane wave in air partially reflects from the surface of a material whose properties are unknown. Measurements of the electric field in the region in front of the interface yield a 1.5 m spacing between maxima, with the first maximum occurring 0.75 m from the interface. A standing wave ratio of 5 is measured. Determine the intrinsic impedance, η_u, of the unknown material.

Solution. The 1.5 m spacing between maxima is $\lambda/2$, implying a wavelength is 3.0 m, or $f = 100$ MHz. The first maximum at 0.75 m is thus at a distance of $\lambda/4$ from the interface, which means that a field minimum occurs at the boundary. Thus Γ will be real and negative. We use (27) to write

$$|\Gamma| = \frac{s-1}{s+1} = \frac{5-1}{5+1} = \frac{2}{3}$$

So

$$\Gamma = -\frac{2}{3} = \frac{\eta_u - \eta_0}{\eta_u + \eta_0}$$

which we solve for η_u to obtain

$$\eta_u = \frac{1}{5}\eta_0 = \frac{377}{5} = 75.4\,\Omega$$

12.3 WAVE REFLECTION FROM MULTIPLE INTERFACES

So far we have treated the reflection of waves at the single boundary that occurs between semi-infinite media. In this section, we consider wave reflection from materials that are finite in extent, such that we must consider the effect of the front and back surfaces. Such a two-interface problem would occur, for example, for light incident on a flat piece of glass. Additional interfaces are present if the glass is coated with one or more layers of dielectric material for the purpose (as we will see) of reducing reflections. Such problems in which more than one interface is involved are frequently encountered; single interface problems are in fact more the exception than the rule.

Consider the general situation shown in Fig. 12.6, in which a uniform plane wave propagating in the forward z direction is normally incident from the left onto the interface between regions 1 and 2; these have intrinsic impedances, η_1 and η_2. A third region of impedance η_3 lies beyond region 2, and so a second interface exists between regions 2 and 3. We let the second interface location occur at $z = 0$, and so all positions to the left will be described by values of z that are negative. The width of the second region is l, so the first interface will occur at position $z = -l$.

When the incident wave reaches the first interface, events occur as follows: A portion of the wave reflects, while the remainder is transmitted, to propagate toward the second interface. There, a portion is transmitted into region 3, while

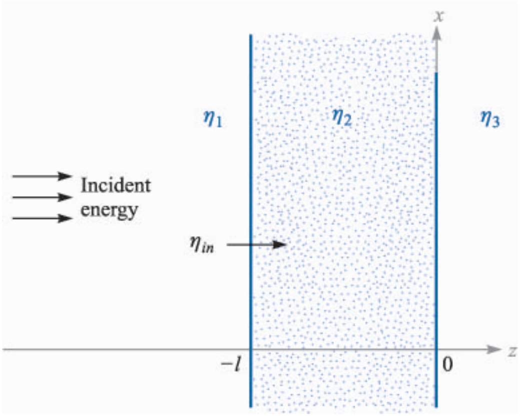

FIGURE 12.6
Basic two-interface problem, in which the impedances of regions 2 and 3, along with the finite thickness of region 2, are accounted for in the input impedance at the front surface, η_{in}.

the rest reflects and returns to the first interface; there it is again partially reflected. This reflected wave then combines with additional transmitted energy from region 1, and the process repeats. We thus have a complicated sequence of multiple reflections that occur within region 2, with partial transmission at each bounce. To analyze the situation in this way would involve keeping track of a very large number of reflections; this would be necessary when studying the *transient* phase of the process, where the incident wave first encounters the interfaces.

If the incident wave is left on for all time, however, a *steady-state* situation is eventually reached, in which: (1) an overall fraction of the incident wave is reflected from the two-interface configuration and back-propagates in region 1 with a definite amplitude and phase; (2) an overall fraction of the incident wave is transmitted through the two interfaces and forward-propagates in the third region; (3) a net backward wave exists in region 2, consisting of all reflected waves from the second interface; (4) a net forward wave exists in region 2, which is the superposition of the transmitted wave through the first interface, and all waves in region 2 that have reflected from the first interface and are now forward-propagating. The effect of combining many co-propagating waves in this way is to establish a single wave which has a definite amplitude and phase, determined through the sums of the amplitudes and phases of all the component waves. In steady state, we thus have a total of five waves to consider. These are the incident and net reflected waves in region 1, the net transmitted wave in region 3, and the two counter-propagating waves in region 2.

Let us assume all regions are composed of lossless media, and consider the two waves in region 2. Taking these as x-polarized, their electric fields add to yield

$$E_{xs2} = E_{x20}^{+}\, e^{-j\beta_2 z} + E_{x20}^{-}\, e^{j\beta_2 z} \tag{28}$$

where $\beta_2 = \omega\sqrt{\epsilon_{R2}}/c$, and where the amplitudes, E_{x20}^{+} and E_{x20}^{-}, are complex. The y-polarized magnetic field is similarly written, using complex amplitudes:

$$H_{ys2} = H_{y20}^{+}\, e^{-j\beta_2 z} + H_{y20}^{-}\, e^{j\beta_2 z} \tag{29}$$

We now note that the forward and backward electric field amplitudes in region 2 are related through the reflection coefficient at the second interface, Γ_{23}, where

$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \tag{30}$$

We thus have

$$E^-_{x20} = \Gamma_{23} E^+_{x20} \tag{31}$$

We then write the magnetic field amplitudes in terms of electric field amplitudes through

$$H^+_{y20} = \frac{1}{\eta_2} E^+_{x20} \tag{32}$$

and

$$H^-_{y20} = -\frac{1}{\eta_2} E^-_{x20} = -\frac{1}{\eta_2} \Gamma_{23} E^+_{x20} \tag{33}$$

We now define the *wave impedance*, η_w, as the z-dependent ratio of the total electric field to the total magnetic field. In region 2, this becomes, using (28) and (29)

$$\eta_w(z) = \frac{E_{xs2}}{H_{ys2}} = \frac{E^+_{x20}\, e^{-j\beta_2 z} + E^-_{x20}\, e^{j\beta_2 z}}{H^+_{y20}\, e^{-j\beta_2 z} + H^-_{y20}\, e^{j\beta_2 z}}$$

Then, using (31), (32), and (33), we obtain

$$\eta_w(z) = \eta_2 \left[\frac{e^{-j\beta_2 z} + \Gamma_{23} e^{j\beta_2 z}}{e^{-j\beta_2 z} - \Gamma_{23} e^{j\beta_2 z}} \right] \tag{34}$$

Now, using (30) and Euler's identity, we have

$$\eta_w(z) = \eta_2 \times \frac{(\eta_3 + \eta_2)(\cos\beta_2 z - j\sin\beta_2 z) + (\eta_3 - \eta_2)(\cos\beta_2 z + j\sin\beta_2 z)}{(\eta_3 + \eta_2)(\cos\beta_2 z - j\sin\beta_2 z) - (\eta_3 - \eta_2)(\cos\beta_2 z + j\sin\beta_2 z)}$$

This is easily simplified to yield

$$\eta_w(z) = \eta_2 \frac{\eta_3 \cos\beta_2 z - j\eta_2 \sin\beta_2 z}{\eta_2 \cos\beta_2 z - j\eta_3 \sin\beta_2 z} \tag{35}$$

We now use the wave impedance in region 2 to solve our reflection problem. Of interest to us is the net reflected wave amplitude at the first interface. Since tangential **E** and **H** are continuous across the boundary, we have

$$E^+_{xs1} + E^-_{xs1} = E_{xs2} \qquad (z = -l) \tag{36}$$

and

$$H^+_{ys1} + H^-_{ys1} = H_{ys2} \qquad (z = -l) \tag{37}$$

Then, in analogy to (7) and (8), we may write

$$E^+_{x10} + E^-_{x10} = E_{xs2}(z = -l) \tag{38}$$

and

$$\frac{E_{x10}^{+}}{\eta_1} - \frac{E_{x10}^{-}}{\eta_1} = \frac{E_{xs2}(z=-l)}{\eta_w(-l)} \tag{39}$$

where E_{x10}^{+} and E_{x10}^{-} are the amplitudes of the incident and reflected fields. We call $\eta_w(-l)$ the *input impedance*, η_{in}, to the two-interface combination. We now solve (38) and (39) together, eliminating E_{xs2}, to obtain

$$\boxed{\frac{E_{x10}^{-}}{E_{x10}^{+}} = \Gamma = \frac{\eta_{in} - \eta_1}{\eta_{in} + \eta_1}} \tag{40}$$

To find the input impedance, we evaluate (35) at $z = -l$, resulting in

$$\boxed{\eta_{in} = \eta_2 \frac{\eta_3 \cos \beta_2 l + j\eta_2 \sin \beta_2 l}{\eta_2 \cos \beta_2 l + j\eta_3 \sin \beta_2 l}} \tag{41}$$

Equations (40) and (41) are general results that enable us to calculate the net reflected wave amplitude and phase from two parallel interfaces between lossless media.[1] Note the dependence on the interface spacing, l, and on the wavelength as measured in region 2, characterized by β_2. Of immediate importance to us is the fraction of the incident power that reflects from the dual interface and back-propagates in region 1. As we found earlier, this fraction will be $|\Gamma|^2$. Also of interest is the transmitted power, which propagates away from the second interface in region 3. It is simply the remaining power fraction, which is $1 - |\Gamma|^2$. The power in region 2 stays constant in steady state; power leaves that region to form the reflected and transmitted waves, but is immediately replenished by the incident wave.

An important result of situations involving two interfaces is that it is possible to achieve total transmission in certain cases. From (40), we see that total transmission occurs when $\Gamma = 0$, or when $\eta_{in} = \eta_1$. In this case we say that the input impedance is *matched* to that of the incident medium. There are a few methods of accomplishing this.

As a start, suppose that $\eta_3 = \eta_1$, and region 2 is of thickness such that $\beta_2 l = m\pi$, where m is an integer. Now $\beta_2 = 2\pi/\lambda_2$, where λ_2 is the wavelength *as measured in region 2*. Therefore

$$\frac{2\pi}{\lambda_2} l = m\pi$$

[1] For convenience, (38) and (39) have been written for a specific time at which the incident wave amplitude, E_{x10}^{+}, occurs at $z = -l$. This establishes a zero-phase reference at the front interface for the incident wave, and so it is from this reference that the reflected wave phase is determined. Equivalently, we have repositioned the $z = 0$ point at the front interface. Eq. (41) allows this, since it is only a function of the interface spacing, l.

or

$$\boxed{l = m\frac{\lambda_2}{2}} \tag{42}$$

With $\beta_2 l = m\pi$, the second region thickness is an integer multiple of half-wavelengths as measured in that medium. Equation (41) now reduces to $\eta_{in} = \eta_3$. *Thus the general effect of a multiple half-wave thickness is to render the second region immaterial to the results on reflection and transmission*. Equivalently, we have a single interface problem involving η_1 and η_3. Now, with $\eta_3 = \eta_1$, we have a matched input impedance, and there is no net reflected wave. This method of choosing the region 2 thickness is known as *half-wave matching*. Its applications include, for example, antenna housings on airplanes known as *radomes*, which form a part of the fuselage. The antenna, inside the aircraft, can transmit and receive through this layer which can be shaped to enable good aerodynamic characteristics. Note that the half-wave matching condition no longer applies as we deviate from the wavelength that satisfies it. When this is done, the device reflectivity increases (with increased wavelength deviation), so it ultimately acts as a bandpass filter.

Another application, typically seen in optics, is the *Fabry-Perot interferometer*. This, in its simplest form, consists of a single block of glass or other transparent material, whose thickness, l, is set to transmit wavelengths which satisfy the condition, $\lambda_2 = 2l/m$. Often, it is desired to transmit only one wavelength, not several, as (42) would allow. We would therefore like to ensure that adjacent wavelengths that are passed through the device are separated as far as possible. This separation is in general given by

$$\lambda_{m-1} - \lambda_m = \Delta\lambda_f = \frac{2l}{m-1} - \frac{2l}{m} = \frac{2l}{m(m-1)} \doteq \frac{2l}{m^2}$$

Note that m is the number of half-wavelengths in region 2, or $m = 2l/\lambda_2$, where λ_2 is the desired wavelength for transmission. Thus

$$\Delta\lambda_f \doteq \frac{\lambda_2^2}{2l} \tag{43}$$

$\Delta\lambda_f$ is known as the *free spectral range* of the Fabry-Perot interferometer. The interferometer can be used as a narrow-band filter (transmitting a desired wavelength and a narrow spectrum around this wavelength) if the spectrum to be filtered is narrower than the free spectral range.

Example 12.4

Suppose we wish to filter an optical spectrum of full-width $\Delta\lambda_s = 50$ nm, and whose center wavelength is in the red part of the visible spectrum at 600 nm, where one nm (nanometer) is 10^{-9} m. A Fabry-Perot filter is to be used, consisting of a lossless glass

plate in air, having relative permittivity $\epsilon'_R = \epsilon_R = 2.1$. We need to find the required range of glass thicknesses such that multiple wavelength orders will not be transmitted.

Solution. We require that the free spectral range be greater than the optical spectral width, or $\Delta\lambda_f > \Delta\lambda_s$. Thus, using (43)

$$l < \frac{\lambda_2^2}{2\Delta\lambda_s}$$

where

$$\lambda_2 = \frac{600}{\sqrt{2.1}} = 414\,\text{nm}$$

So

$$l < \frac{414^2}{2(50)} = 1.7 \times 10^3\,\text{nm} = 1.7\,\mu\text{m}$$

where 1μm (micrometer) $= 10^{-6}$ m. Fabricating a glass plate of this thickness or less is somewhat ridiculous to contemplate. Instead, what is often used is an air space of thickness on this order, between two thick plates whose surfaces on the sides opposite the air space are antireflection coated. This is in fact a more versatile configuration since the wavelength to be transmitted (and the free spectral range) can be adjusted by varying the plate separation.

Next we remove the restriction $\eta_1 = \eta_3$ and look for a way to produce zero reflection. Suppose we set $\beta_2 l = (2m-1)\pi/2$, or an odd multiple of $\pi/2$. This means that

$$\frac{2\pi}{\lambda_2} l = (2m-1)\frac{\pi}{2} \qquad (m = 1, 2, 3, \ldots)$$

or

$$\boxed{l = (2m-1)\frac{\lambda_2}{4}} \tag{44}$$

The thickness is an odd multiple of a quarter wavelength as measured in region 2. Under this condition (41) reduces to

$$\eta_{in} = \frac{\eta_2^2}{\eta_3} \tag{45}$$

Typically, we choose the second region impedance to allow matching between given impedances η_1 and η_3. To achieve total transmission, we require that $\eta_{in} = \eta_1$, so that the required second region impedance becomes

$$\boxed{\eta_2 = \sqrt{\eta_1 \eta_3}} \tag{46}$$

With the conditions given by (44) and (46) satisfied, we have performed *quarter-wave matching*. The design of anti reflective coatings for optical devices is based on this principle.

Example 12.5

We wish to coat a glass surface with an appropriate dielectric layer to provide total transmission from air to the glass at a wavelength of 570 nm. The glass has dielectric constant, $\epsilon_R = 2.1$. Determine the required dielectric constant for the coating and its minimum thickness.

Solution. The known impedances are $\eta_1 = 377\,\Omega$ and $\eta_3 = 377/\sqrt{2.1} = 260\,\Omega$. Using (46) we have

$$\eta_2 = \sqrt{(377)(260)} = 313\,\Omega$$

The dielectric constant of region 2 will then be

$$\epsilon_{R2} = \left(\frac{377}{313}\right)^2 = 1.45$$

The wavelength in region 2 will be

$$\lambda_2 = \frac{570}{\sqrt{1.45}} = 473\,\text{nm}$$

The minimum thickness of the dielectric layer is then

$$l = \frac{\lambda_2}{4} = 118\,\text{nm} = 0.118\,\mu\text{m}$$

The procedure in this section for evaluating wave reflection has involved calculating an effective impedance at the first interface, η_{in}, which is expressed in terms of the impedances that lie beyond the front surface. This process of *impedance transformation* is more apparent when we consider problems involving more than two interfaces.

For example, consider the three-interface situation shown in Fig. 12.7, where a wave is incident from the left in region 1. We wish to determine the fraction of the incident power that is reflected and back-propagates in region 1, and the fraction of the incident power that is transmitted into region 4. To do this, we need to find the input impedance at the front surface (the interface between regions 1 and 2). We start by transforming the impedance of region 4 to form the input impedance at the boundary between regions 2 and 3. This is shown as $\eta_{in,b}$ in the figure. Using (41), we have

$$\eta_{in,b} = \eta_3 \frac{\eta_4 \cos\beta_3 l_b + j\eta_3 \sin\beta_3 l_b}{\eta_3 \cos\beta_3 l_b + j\eta_4 \sin\beta_3 l_b} \tag{47}$$

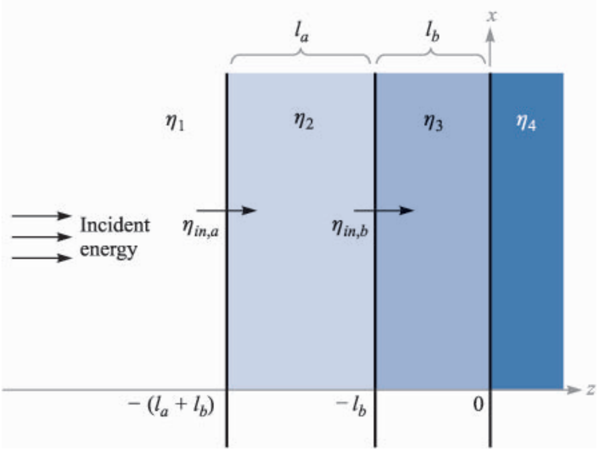

FIGURE 12.7
A three-interface problem, in which input impedance $\eta_{in,b}$ is transformed back to the front interface to form input impedance $\eta_{in,a}$.

We have now effectively reduced the situation to a two-interface problem in which $\eta_{in,b}$ is the impedance of all that lies beyond the second interface. The input impedance at the front interface, $\eta_{in,a}$, is now found by transforming $\eta_{in,b}$ as follows:

$$\eta_{in,a} = \eta_2 \frac{\eta_{in,b} \cos \beta_2 l_a + j\eta_2 \sin \beta_2 l_a}{\eta_2 \cos \beta_2 l_a + j\eta_{in,b} \sin \beta_2 l_a} \tag{48}$$

The reflected power fraction is now $|\Gamma|^2$, where

$$\Gamma = \frac{\eta_{in,a} - \eta_1}{\eta_{in,a} + \eta_1}$$

The fraction of the power transmitted into region 4 is, as before, $1 - |\Gamma|^2$. The method of impedance transformation can be applied in this manner to any number of interfaces. The process, although tedious, is easily handled by a computer.

The motivation for using multiple layers to reduce reflection is that the resulting structure is less sensitive to deviations from the design wavelength if the impedances are arranged to progressively increase or decrease from layer to layer. In using multiple layers to antireflection coat a camera lens, for example, the layer on the lens surface would be of impedance very close to that of the glass. Subsequent layers are given progressively higher impedances. With a large number of layers fabricated in this way, the situation begins to approach (but never reaches) the ideal case, in which the top layer impedance matches that of air, while the impedances of deeper layers continuously decrease until reaching the value of the glass surface. With this continuously varying impedance, there would be no surface from which to reflect, and so light of any wavelength is totally transmitted. Multilayer coatings designed in this way produce excellent broadband transmission characteristics.

The impedance transformation method for handling multiple interfaces applies not only to plane waves at boundaries, but also to loaded transmission lines of finite length, and to cascaded transmission lines. We will encounter problems of this type in the next chapter, which we will solve using exactly the same mathematics.

D12.3. A uniform plane wave in air is normally incident on a dielectric slab of thickness $\lambda_2/4$, and intrinsic impedance $\eta_2 = 260\ \Omega$. Determine the magnitude and phase of the reflection coefficient.

Ans. 0.356; 180°.

12.4 PLANE WAVE PROPAGATION IN GENERAL DIRECTIONS

In this section we will learn how to mathematically describe uniform plane waves that propagate in any direction. Our motivation for doing this is our need to address the problem of incident waves on boundaries that are not perpendicular to the propagation direction. Such problems of *oblique incidence* generally occur, with normal incidence being a special case. Addressing such problems requires (as always) that we establish an appropriate coordinate system. With the boundary positioned in the x, y plane, for example, the incident wave will propagate in a direction that could involve all three coordinate axes, whereas with normal incidence, we were only concerned with propagation along z. We need a mathematical formalism that will allow for the general direction case.

Let us consider a wave that propagates in a lossless medium, with propagation constant $\beta = k = \omega\sqrt{\mu\epsilon}$. For simplicity, we consider a two-dimensional case, where the wave travels in a direction between the x and z axes. The first step is to consider the propagation constant as a *vector*, $\mathbf{k}$, indicated in Fig. 12.8. The direction of $\mathbf{k}$ is the propagation direction, which is the same as the direction of the Poynting vector in our case.[2] The magnitude of $\mathbf{k}$ is the phase shift per unit distance *along that direction*. Part of the process of characterizing a wave involves specifying its phase at any spatial location. For the waves we have considered that propagate along the z axis, this was accomplished by the factor $e^{\pm jkz}$ in the phasor form. To specify the phase in our two-dimensional problem, we make use of the vector nature of $\mathbf{k}$, and consider the phase at a general location, (x, z), described through the position vector $\mathbf{r}$. The phase at that location, referenced to the origin, is given by the projection of $\mathbf{k}$ along $\mathbf{r}$ times the magnitude of $\mathbf{r}$, or just

[2] We assume here that the wave is in an isotropic medium, where the permittivity and permeability do not change with field orientation. In anisotropic media (where ϵ and/or μ depend on field orientation), the directions of the Poynting vector and $\mathbf{k}$ may differ.

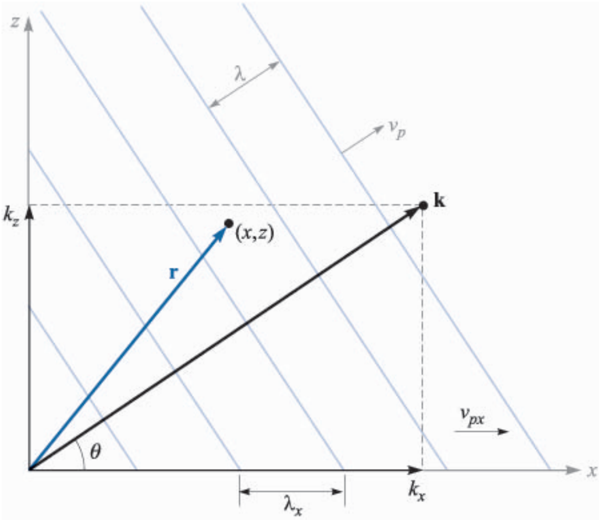

FIGURE 12.8 Representation of a uniform plane wave with wavevector **k** at angle θ to the x axis. The phase at point (x, z) is given by $\mathbf{k} \cdot \mathbf{r}$. Planes of constant phase (shown as lines perpendicular to **k**) are spaced by wavelength λ, but have wider spacing when measured along the x or z axes.

$\mathbf{k} \cdot \mathbf{r}$. If the electric field is of magnitude E_0, we can thus write down the phasor form of the wave in Fig. 12.8 as

$$\boxed{\mathbf{E}_s = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}} \tag{49}$$

The minus sign in the exponent indicates that the phase along **r** moves in time in the direction of increasing **r**. Again, the wave power flow in an isotropic medium occurs in the direction along which the phase shift per unit distance is maximum—or along **k**. The vector **r** serves as a means to measure phase at any point using **k**. This construction is easily extended to three dimensions by allowing **k** and **r** to each have three components.

In our two-dimensional case of Fig. 12.8, we can express **k** in terms of its x and z components:

$$\boxed{\mathbf{k} = k_x \mathbf{a_x} + k_z \mathbf{a_z}}$$

The position vector, **r**, can be similarly expressed:

$$\boxed{\mathbf{r} = x\,\mathbf{a_x} + z\,\mathbf{a_z}}$$

so that

$$\mathbf{k} \cdot \mathbf{r} = k_x x + k_z z$$

Equation (49) now becomes

$$\mathbf{E}_s = \mathbf{E}_0 e^{-j(k_x x + k_z z)} \tag{50}$$

Whereas Eq. (49) provided the general form of the wave, Eq. (50) is the form that is specific to the situation. Given a wave expressed by (50), the angle of propagation from the x axis is readily found through

$$\theta = \tan^{-1}\left(\frac{k_z}{k_x}\right)$$

The wavelength and phase velocity depend on the direction one is considering. In the direction of $\mathbf{k}$, these will be

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{(k_x^2 + k_z^2)^{1/2}}$$

and

$$v_p = \frac{\omega}{k} = \frac{\omega}{(k_x^2 + k_z^2)^{1/2}}$$

If, for example, we consider the x direction, these quantities will be

$$\lambda_x = \frac{2\pi}{k_x}$$

and

$$v_{px} = \frac{\omega}{k_x}$$

Note that both λ_x and v_{px} are greater than their counterparts along the direction of $\mathbf{k}$. This result, at first surprising, can be understood through the geometry of Fig. 12.8. The diagram shows a series of phase fronts (planes of constant phase), which intersect $\mathbf{k}$ at right angles. The phase shift between adjacent fronts is set at 2π in the figure; this corresponds to a spatial separation along the $\mathbf{k}$ direction of one wavelength, as shown. The phase fronts intersect the x axis, and we see that *along* x the front separation is greater than it was along $\mathbf{k}$. λ_x is the spacing between fronts along x, and is indicated on the figure. The phase velocity along x is the velocity of the intersection points between the phase fronts and the x axis. Again, from the geometry, we see that this velocity must be faster than the velocity along $\mathbf{k}$, and will of course exceed the speed of light in the medium. This does not constitute a violation of special relativity, however, since the energy in the wave flows in the direction of $\mathbf{k}$, and not along x or z. The wave frequency is $f = \omega/2\pi$, and is invariant with direction. Note, for example, that in the directions we have considered,

$$f = \frac{v_p}{\lambda} = \frac{v_{px}}{\lambda_x} = \frac{\omega}{2\pi}$$

Example 12.6

Consider a 50 MHz uniform plane wave having electric field amplitude 10 V/m. The medium is lossless, having $\epsilon_R = \epsilon'_R = 9.0$ and $\mu_R = 1.0$. The wave propagates in the x, y plane at a 30° angle to the x axis, and is linearly polarized along z. Write down the phasor expression for the electric field.

Solution. The propagation constant magnitude is

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega\sqrt{\epsilon_R}}{c} = \frac{2\pi \times 50 \times 10^6(3)}{3 \times 10^8} = 3.14\,\mathrm{m}^{-1}$$

The vector $\mathbf{k}$ is now

$$\mathbf{k} = 3.14(\cos 30\,\mathbf{a_x} + \sin 30\,\mathbf{a_y}) = 2.7\mathbf{a_x} + 1.6\mathbf{a_y}\,\mathrm{m}^{-1}$$

Then

$$\mathbf{r} = x\,\mathbf{a_x} + y\,\mathbf{a_y}$$

With the electric field directed along z, the phasor form becomes

$$\mathbf{E_s} = E_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\,\mathbf{a_z} = 10e^{-j(2.7x+1.6y)}\,\mathbf{a_z}$$

✔ **D12.4.** For Example 12.6, calculate λ_x, λ_y, v_{px}, and v_{py}.

Ans. 2.3 m; 3.9 m; 1.2×10^8 m/s; 2.0×10^8 m/s.

12.5 PLANE WAVE REFLECTION AT OBLIQUE INCIDENCE ANGLES

We now consider the problem of wave reflection from plane interfaces, in which the incident wave propagates at some angle to the surface. Our objectives are (1) to determine the relation between incident, reflected, and transmitted angles, and (2) to derive reflection and transmission coefficients that are functions of the incident angle and wave polarization. We will also show that cases exist in which total reflection or total transmission may occur at the interface between two dielectrics if the angle of incidence and the polarization are appropriately chosen.

The situation is illustrated in Fig. 12.9, in which the incident wave direction and position-dependent phase are characterized by wavevector, $\mathbf{k}_1^+$. The angle of incidence is the angle between $\mathbf{k}_1^+$ and a line that is normal to the surface (the x axis in this case). The incidence angle is shown as θ_1. The reflected wave, characterized by wavevector $\mathbf{k}_1^-$, will propagate away from the interface at angle θ'_1. Finally, the transmitted wave, characterized by $\mathbf{k}_2$, will propagate into the second region at angle θ_2 as shown. One would suspect (from previous experience) that the incident and reflected angles are equal ($\theta_1 = \theta'_1$), which is correct. We need to show this, however, to be complete.

The two media are lossless dielectrics, characterized by intrinsic impedances, η_1 and η_2. We will assume, as before, that the materials are non-

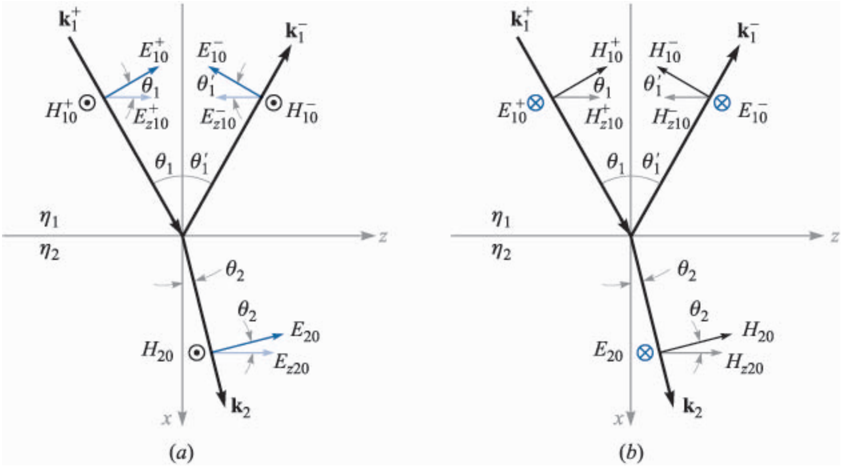

FIGURE 12.9
Geometries for plane wave incidence at angle θ_1 onto an interface between dielectrics having intrinsic impedances η_1 and η_2. The two polarization cases are shown: (*a*) p-polarization (or TM), with **E** in the plane of incidence; (*b*) s-polarization (or TE), with **E** perpendicular to the plane of incidence.

magnetic, and thus have permeability μ_0. Consequently, the materials are adequately described by specifying their dielectric constants, ϵ_{R1} and ϵ_{R2}. It is convenient at this stage to introduce the *refractive index* of the material, n, defined as the square root of the dielectric constant. We thus have $n_1 = \sqrt{\epsilon_{R1}}$ and $n_2 = \sqrt{\epsilon_{R2}}$. Use of either parameter (index or dielectric constant) is acceptable, with refractive index used almost exclusively in the optical frequency range (on the order of 10^{14} Hz); at lower frequencies, problems are typically posed using dielectric constants. It is important not to confuse index, n, with the similar-appearing Greek η (intrinsic impedance), which has an entirely different meaning.

In Fig. 12.9, two cases are shown which differ by the choice of electric field orientation. In Fig. 12.9*a*, the **E** field is polarized in the plane of the page, with **H** therefore perpendicular to the page and pointing outward. In this illustration, the plane of the page is also the *plane of incidence*, which is more precisely defined as the plane spanned by the incident **k** vector and the normal to the surface. With **E** lying in the plane of incidence, the wave is said to have *parallel polarization*, or is *p-polarized* (**E** is parallel to the incidence plane). Note that while **H** is perpendicular to the incidence plane, it lies parallel to the interface, or transverse to the direction normal to the interface. Consequently, another name for this type of polarization is *transverse magnetic*, or TM polarization.

Fig. 12.9*b* shows the situation in which the field directions have been rotated by 90°. Now **H** lies in the plane of incidence, whereas **E** is perpendicular to the plane. Since **E** is used to define polarization, the configuration is called

perpendicular polarization, or is *s-polarized*.[3] **E** is also parallel to the interface, and so the case is also called *transverse electric*, or TE polarization. We will find that the reflection and transmission coefficients will differ for the two polarization types, but that reflection and transmission angles will not depend on polarization. We only need to consider s and p polarizations, since any other field direction can be constructed as some combination of s and p waves.

Our desired knowledge of reflection and transmission coefficients, as well as how the angles relate, can be found through the field boundary conditions at the interface. Specifically, we require that the transverse components of **E** and **H** be continuous across the interface. These were the conditions we used to find Γ and τ for normal incidence ($\theta_1 = 0$), which is in fact a special case of our current problem. We will consider the case of p-polarization (Fig. 12.9*a*) first. To begin, we write down the incident, reflected, and transmitted fields in phasor form, using the notation developed in the previous section:

$$\mathbf{E}_{s1}^{+} = \mathbf{E}_{10}^{+}\, e^{-j\mathbf{k}_1^{+}\cdot\mathbf{r}} \tag{51}$$

$$\mathbf{E}_{s1}^{-} = \mathbf{E}_{10}^{-}\, e^{-j\mathbf{k}_1^{-}\cdot\mathbf{r}} \tag{52}$$

$$\mathbf{E}_{s2} = \mathbf{E}_{20}\, e^{-j\mathbf{k}_2\cdot\mathbf{r}} \tag{53}$$

where

$$\mathbf{k}_1^{+} = k_1(\cos\theta_1\, \mathbf{a_x} + \sin\theta_1\, \mathbf{a_z}) \tag{54}$$

$$\mathbf{k}_1^{-} = k_1\left(-\cos\theta_1'\, \mathbf{a_x} + \sin\theta_1'\, \mathbf{a_z}\right) \tag{55}$$

$$\mathbf{k}_2 = k_2(\cos\theta_2\, \mathbf{a_x} + \sin\theta_2\, \mathbf{a_z}) \tag{56}$$

and where

$$\mathbf{r} = x\,\mathbf{a_x} + z\,\mathbf{a_z} \tag{57}$$

The wavevector magnitudes are $k_1 = \omega\sqrt{\epsilon_{R1}}/c = n_1\omega/c$ and $k_2 = \omega\sqrt{\epsilon_{R2}}/c = n_2\omega/c$.

Now, to evaluate the boundary condition that requires continuous tangential electric field, we need to find the components of the electric fields (z components) that are parallel to the interface. Projecting all **E** fields in the z direction, and using (51) through (57), we find

$$E_{zs1}^{+} = E_{z10}^{+}\, e^{-j\mathbf{k}_1^{+}\cdot\mathbf{r}} = E_{10}^{+}\cos\theta_1 e^{-jk_1(x\cos\theta_1 + z\sin\theta_1)} \tag{58}$$

$$E_{zs1}^{-} = E_{z10}^{-}\, e^{-j\mathbf{k}_1^{-}\cdot\mathbf{r}} = E_{10}^{-}\cos\theta_1' e^{jk_1(x\cos\theta_1' - z\sin\theta_1')} \tag{59}$$

$$E_{zs2} = E_{z20}\, e^{-j\mathbf{k}_2\cdot\mathbf{r}} = E_{20}\cos\theta_2 e^{-jk_2(x\cos\theta_2 + z\sin\theta_2)} \tag{60}$$

[3] The s designation is an abbreviation for the German *senkrecht*, meaning *perpendicular*. The p in p-polarized is an abbreviation for the German word for parallel, which is *parallel*.

The boundary condition for continuous tangential electric field now reads:

$$E^+_{zs1} + E^-_{zs1} = E_{zs2} \quad (\text{at } x = 0)$$

We now substitute Eqs. (58) through (60) into (61), and evaluate the result at $x = 0$ to obtain:

$$E^+_{10}\cos\theta_1\, e^{-jk_1 z\sin\theta_1} + E^-_{10}\cos\theta_1'\, e^{-jk_1 z\sin\theta_1'} = E_{20}\cos\theta_2\, e^{-jk_2 z\sin\theta_2} \tag{61}$$

Note that E^+_{10}, E^-_{10}, and E_{20} are all constants (independent of z). Further, we require that (61) hold for all values of z (everywhere on the interface). For this to occur, it must follow that all the phase terms appearing in (61) are equal. Specifically,

$$k_1 z\sin\theta_1 = k_1 z\sin\theta_1' = k_2 z\sin\theta_2$$

From this, we see immediately that $\theta_1' = \theta_1$, or the angle of reflection is equal to the angle of incidence. We also find that

$$\boxed{k_1\sin\theta_1 = k_2\sin\theta_2} \tag{62}$$

Eq. (62) is known as *Snell's law of refraction*. Since in general, $k = n\omega/c$, we can rewrite (62) in terms of the refractive indices:

$$\boxed{n_1\sin\theta_1 = n_2\sin\theta_2} \tag{63}$$

Eq. (63) is the form of Snell's law that is most readily used for our present case of nonmagnetic dielectrics. Eq. (62) is a more general form which would apply, for example, to cases involving materials with different permeabilities as well as different permittivities. In general, we would have $k_1 = (\omega/c)\sqrt{\mu_{R1}\epsilon_{R1}}$ and $k_2 = (\omega/c)\sqrt{\mu_{R2}\epsilon_{R2}}$.

Having found the relations between angles, we next turn to our second objective, which is to determine the relations between the amplitudes, E^+_{10}, E^-_{10}, and E_{20}. To accomplish this, we need to consider the other boundary condition, requiring tangential continuity of **H** at $x = 0$. The magnetic field vectors for the p-polarized wave are all negative y-directed. At the boundary the field amplitudes are related through

$$H^+_{10} + H^-_{10} = H_{20} \tag{64}$$

Then, using the fact that $\theta_1' = \theta_1$, and invoking Snell's law, (61) becomes

$$E^+_{10}\cos\theta_1 + E^-_{10}\cos\theta_1 = E_{20}\cos\theta_2 \tag{65}$$

Using the medium intrinsic impedances, we know for example, that $E^+_{10}/H^+_{10} = \eta_1$ and $E^+_{20}/H^+_{20} = \eta_2$. Eq. (64) can be written as follows:

$$\frac{E_{10}^{+}\cos\theta_1}{\eta_{1p}} - \frac{E_{10}^{-}\cos\theta_1}{\eta_{1p}} = \frac{E_{20}^{+}\cos\theta_2}{\eta_{2p}} \tag{66}$$

Note the minus sign in front of the second term in (66), which results from the fact that $E_{10}^{-}\cos\theta_1$ is negative (from Fig. 12.9*a*), whereas H_{10}^{-} is positive (again from the figure). In writing Eq. (66), *effective impedances*, valid for p-polarization, are defined through

$$\boxed{\eta_{1p} = \eta_1 \cos\theta_1} \tag{67}$$

and

$$\boxed{\eta_{2p} = \eta_2 \cos\theta_2} \tag{68}$$

Using this representation, Eqs. (65) and (66) are now in a form that enables them to be solved together for the ratios E_{10}^{-}/E_{10}^{+} and E_{20}/E_{10}^{+}. Performing analogous procedures to those used in solving (7) and (8), we find the reflection and transmission coefficients:

$$\boxed{\Gamma_p = \frac{E_{10}^{-}}{E_{10}^{+}} = \frac{\eta_{2p} - \eta_{1p}}{\eta_{2p} + \eta_{1p}}} \tag{69}$$

$$\boxed{\tau_p = \frac{E_{20}}{E_{10}^{+}} = \frac{2\eta_{2p}}{\eta_{2p} + \eta_{1p}}} \tag{70}$$

A similar procedure can be carried out for s-polarization, referring to Fig. 12.9*b*. The details are left as an exercise; the results are

$$\boxed{\Gamma_s = \frac{E_{y10}^{-}}{E_{y10}^{+}} = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}}} \tag{71}$$

$$\boxed{\tau_s = \frac{E_{y20}}{E_{y10}^{+}} = \frac{2\eta_{2s}}{\eta_{2s} + \eta_{1s}}} \tag{72}$$

where the effective impedances for s-polarization are

$$\boxed{\eta_{1s} = \eta_1 \sec\theta_1} \tag{73}$$

and

$$\boxed{\eta_{2s} = \eta_2 \sec\theta_2} \tag{74}$$

Equations (67) through (74) are what we need to calculate wave reflection and transmission for either polarization, and at any incident angle.

Example 12.7

A uniform plane wave is incident from air onto glass at an angle from the normal of 30°. Determine the fraction of the incident power that is reflected and transmitted for (a) p-polarization and (b) s-polarization. Glass has refractive index $n_2 = 1.45$.

Solution. First, we apply Snell's law to find the transmission angle. Using $n_1 = 1$ for air, we use (63) to find

$$\theta_2 = \sin^{-1}\left(\frac{\sin 30}{1.45}\right) = 20.2^\circ$$

Now, for p-polarization:

$$\eta_{1p} = \eta_1 \cos 30 = (377)(.866) = 326\,\Omega$$

$$\eta_{2p} = \eta_2 \cos 20.2 = \frac{377}{1.45}(.938) = 244\,\Omega$$

Then, using (69), we find

$$\Gamma_p = \frac{244 - 326}{244 + 326} = -0.144$$

The fraction of the incident power that is reflected is

$$\frac{P_r}{P_{inc}} = |\Gamma_p|^2 = .021$$

The transmitted fraction is then

$$\frac{P_t}{P_{inc}} = 1 - |\Gamma_p|^2 = .979$$

For s-polarization, we have

$$\eta_{1s} = \eta_1 \sec 30 = 377/.866 = 435\,\Omega$$

$$\eta_{2s} = \eta_2 \sec 20.2 = \frac{377}{1.45(.938)} = 277\,\Omega$$

Then, using (71):

$$\Gamma_s = \frac{277 - 435}{277 + 435} = -.222$$

The reflected power fraction is thus

$$|\Gamma_s|^2 = .049$$

The fraction of the incident power that is transmitted is

$$1 - |\Gamma_s|^2 = .951$$

In the previous example, reflection coefficient values for the two polarizations were found to be negative. The meaning of a negative reflection coefficient is that the component of the reflected electric field that is parallel to the interface will be directed opposite the incident field component when both are evaluated at the boundary.

The above effect is also observed when the second medium is a perfect conductor. In this case, we know that the electric field inside the conductor must be zero. Consequently, $\eta_2 = E_{20}/H_{20} = 0$, and the reflection coefficients will be $\Gamma_p = \Gamma_s = -1$. Total reflection occurs, regardless of the incident angle or polarization.

Now that we have methods available to us for solving problems involving oblique incidence reflection and transmission, we can explore the special cases of *total reflection* and *total transmission*. We look for special combinations of media, incidence angles, and polarizations that produce these properties. To begin, we identify the necessary condition for total reflection. We want total *power* reflection, so that $|\Gamma|^2 = \Gamma\Gamma^* = 1$, where Γ is either Γ_p or Γ_s. The fact that this condition involves the possibility of a complex Γ allows some flexibility. For the incident medium, we note that η_{1p} and η_{1s} will always be real and positive. On the other hand, when we consider the second medium, η_{2p} and η_{2s} involve factors of $\cos\theta_2$ or $1/\cos\theta_2$, where

$$\cos\theta_2 = \left[1 - \sin^2\theta_2\right]^{1/2} = \left[1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1\right]^{1/2} \tag{75}$$

where Snell's law has been used. We observe that $\cos\theta_2$, and hence η_{2p} and η_{2s}, become imaginary whenever $\sin\theta_1 > n_2/n_1$. Let us consider parallel polarization for example. Under conditions of imaginary η_{2p}, (69) becomes

$$\Gamma_p = \frac{j|\eta_{2p}| - \eta_{1p}}{j|\eta_{2p}| + \eta_{1p}} = -\frac{\eta_{1p} - j|\eta_{2p}|}{\eta_{1p} + j|\eta_{2p}|} = -\frac{Z}{Z^*}$$

where $Z = \eta_{1p} - j|\eta_{2p}|$. *We can therefore see that* $\Gamma_p\Gamma_p^* = 1$, *meaning total power reflection, whenever* η_{2p} *is imaginary*. The same will be true whenever η_{2p} is zero, which will occur when $\sin\theta_1 = n_2/n_1$. We thus have our condition for total reflection, which is

$$\boxed{\sin\theta_1 \geq \frac{n_2}{n_1}} \tag{76}$$

From this condition arises the *critical angle* of total reflection, θ_c, defined through

$$\boxed{\sin\theta_c = \frac{n_2}{n_1}} \tag{77}$$

The total reflection condition can thus be more succinctly written as

$$\boxed{\theta_1 \geq \theta_c \qquad \text{(for total reflection)}} \tag{78}$$

Note that for (76) and (77) to make sense, it must be true that $n_2 < n_1$, or the wave must be incident from a medium of higher refractive index than that of the medium beyond the boundary. For this reason, the total reflection condition is sometimes called total *internal* reflection; it is often seen (and applied) in optical devices such as beam-steering prisms, where light within the glass structure totally reflects from glass-air interfaces.

Example 12.8

A prism is to be used to turn a beam of light by 90°, as shown in Fig. 12.10. Light enters and exits the prism through two antireflective (AR-coated) surfaces. Total reflection is to occur at the back surface, where the incident angle is 45° to the normal. Determine the minimum required refractive index of the prism material if the surrounding region is air.

Solution. Considering the back surface, the medium beyond the interface is air, with $n_2 = 1.00$. Since $\theta_1 = 45°$, (76) is used to obtain

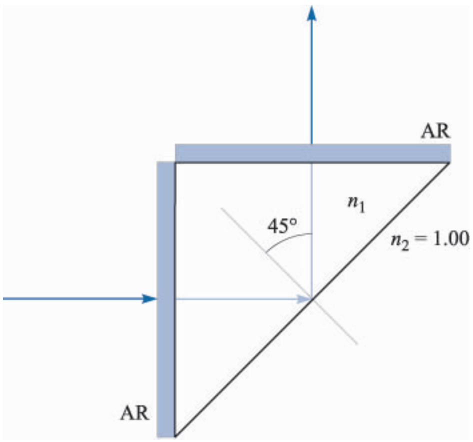

FIGURE 12.10
Beam-steering prism for Example 12.8.

$$n_1 \geq \frac{n_2}{\sin 45} = \sqrt{2} = 1.41$$

Since fused silica glass has refractive index $n_g = 1.45$, it is a suitable material for this application, and is in fact widely used.

Another important application of total reflection is in *optical waveguides*. These, in their simplest form, are constructed of three layers of glass, in which the middle layer has a slightly higher refractive index than the outer two. Fig. 12.11 shows the basic structure. Light, propagating from left to right, is confined to the middle layer by total reflection at the two interfaces, as shown. Optical fiber waveguides are constructed on this principle, in which a cylindrical glass core region of small radius is surrounded coaxially by a lower-index cladding glass material of larger radius. Basic waveguiding principles as applied to metallic and dielectric structures will be presented in Chapter 14.

We next consider the possibility of *total transmission*. In this case the requirement is simply that $\Gamma = 0$. We investigate this possibility for the two polarizations. First, we consider s-polarization. If $\Gamma_s = 0$, then from (71) we require that $\eta_{2s} = \eta_{1s}$, or

$$\eta_2 \sec\theta_2 = \eta_1 \sec\theta_1$$

Using Snell's law to write θ_2 in terms of θ_1, the above becomes

$$\eta_2 \left[1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1\right]^{-1/2} = \eta_1 \left[1 - \sin^2\theta_1\right]^{-1/2}$$

There is no value of θ_1 that will satisfy the above, so we turn instead to p-polarization. Using (67), (68), and (69), with Snell's law, the condition for $\Gamma_p = 0$ is

$$\eta_2 \left[1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_1\right]^{1/2} = \eta_1 \left[1 - \sin^2\theta_1\right]^{1/2}$$

This equation does have a solution, which is

$$\boxed{\sin\theta_1 = \sin\theta_B = \frac{n_2}{\sqrt{n_1^2 + n_2^2}}} \tag{79}$$

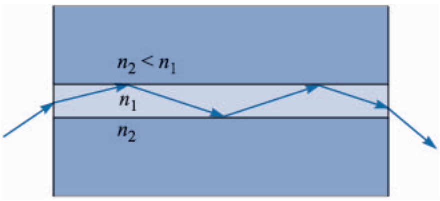

FIGURE 12.11
A dielectric slab waveguide (symmetric case), showing light confinement to the center material by total reflection.

where we have used $\eta_1 = \eta_0/n_1$ and $\eta_2 = \eta_0/n_2$. We call this special angle, θ_B, where total transmission occurs, the *Brewster angle* or *polarization angle*. The latter name comes from the fact that if light having both s- and p-polarization components is incident at $\theta_1 = \theta_B$, the p component will be totally transmitted, leaving the partially reflected light entirely s-polarized. At angles that are slightly off the Brewster angle, the reflected light is still predominantly s-polarized. Most reflected light that we see originates from horizontal surfaces (such as the surface of the ocean), and as such, the light is mostly of horizontal polarization. Polaroid sunglasses take advantage of this fact to reduce glare, since they are made to block transmission of horizontally polarized light, while passing light that is vertically polarized.

Example 12.9

Light is incident from air to glass at Brewster's angle. Determine the incident and transmitted angles.

Solution. Since glass has refractive index $n_2 = 1.45$, the incident angle will be

$$\theta_1 = \theta_B = \sin^{-1}\left(\frac{n_2}{\sqrt{n_1^2 + n_2^2}}\right) = \sin^{-1}\left(\frac{1.45}{\sqrt{1.45^2 + 1}}\right) = 55.4^\circ$$

The transmitted angle is found from Snell's law, through

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_B\right) = \sin^{-1}\left(\frac{n_1}{\sqrt{n_1^2 + n_2^2}}\right) = 34.6^\circ$$

Note from this exercise that $\sin\theta_2 = \cos\theta_B$, which means that the sum of the incident and refracted angles at the Brewster condition is always 90°.

✔ **D12.5.** In Example 12.9, calculate the reflection coefficient for s-polarized light.

Ans. -0.355

Many of the results we have seen in this section are summarized in Fig. 12.12, in which Γ_p and Γ_s, from (69) and (71), are plotted as functions of the incident angle, θ_1. Curves are shown for selected values of the refractive index ratio, n_1/n_2. For all plots in which $n_1/n_2 > 1$, Γ_s and Γ_p achieve a value of ± 1 at the critical angle. At larger angles, the reflection coefficients become imaginary (and are not shown) but nevertheless retain magnitudes of unity. The occurrence of the Brewster angle is evident in the curves for Γ_p (Fig. 12.12*a*), as all curves cross the θ_1 axis. This behavior is not seen in the Γ_s functions (Fig. 12.12*b*), as Γ_s is positive for all values of θ_1 when $n_1/n_2 > 1$, and is negative for $n_1/n_2 < 1$.

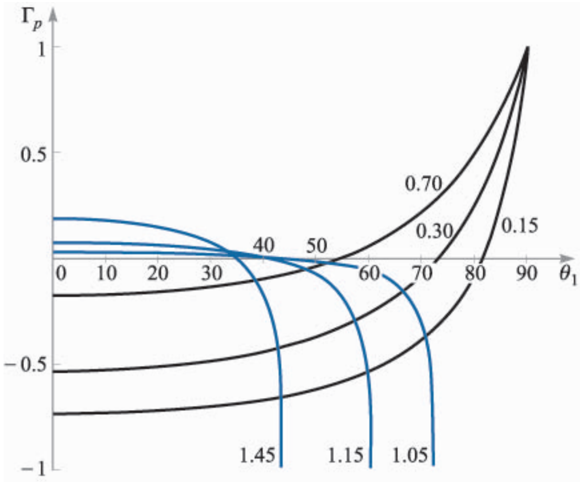

FIGURE 12.12*a*
Plots of Γ_p (Eq. (69)) as functions of the incident angle, θ_1, as per Fig. 12.9*a*. Curves are shown for selected values of the refractive index ratio, n_1/n_2. Both media are lossless and have $\mu_R = 1$. Thus $\eta_1 = \eta_0/n_1$ and $\eta_2 = \eta_0/n_2$.

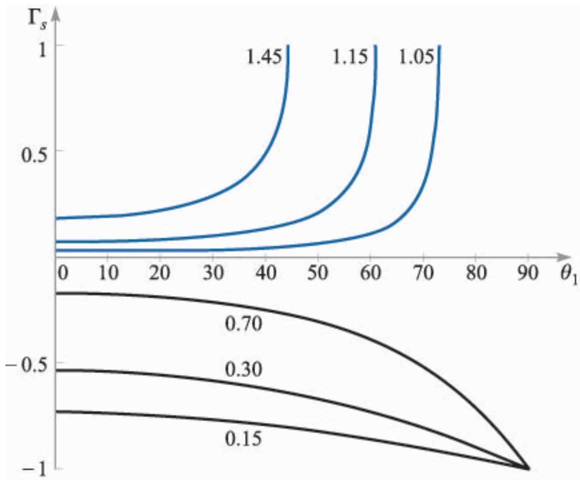

FIGURE 12.12*b*
Plots of Γ_s (Eq. (71)) as functions of the incident angle, θ_1, as per Fig. 12.9*b*. As in Fig. 12.12*a*, the media are lossless, and curves are shown for selected n_1/n_2.

12.6 WAVE PROPAGATION IN DISPERSIVE MEDIA

In Chapter 11, we encountered situations in which the complex permittivity of the medium depends on frequency. This is true in all materials through a number of possible mechanisms. One of these, mentioned earlier, is that oscillating bound charges in a material are in fact harmonic oscillators that have resonant frequencies associated with them (see Appendix D). When the frequency of an incoming electromagnetic wave is at or near a bound charge resonance, the wave will induce strong oscillations; these in turn have the effect of depleting energy from the wave in its original form. The wave thus experiences absorption, and

does so to a greater extent than it would at a frequency that is detuned from resonance. A related effect is that the real part of the dielectric constant will be different at frequencies near resonance than at frequencies far from resonance. In short, resonance effects give rise to values of ϵ' and ϵ'' that will vary continuously with frequency. These in turn will produce a fairly complicated frequency dependence in the attenuation and propagation constants as expressed in Eqs. (35) and (36) in Chapter 11.

This section concerns the effect of a frequency-varying dielectric constant (or refractive index) on a wave as it propagates in an otherwise lossless medium. This situation in fact arises quite often, since significant refractive index variation can occur at frequencies far away from resonance, where absorptive losses are negligible. A classic example of this is the separation of white light into its component colors by a glass prism. In this case, the frequency-dependent refractive index results in different angles of refraction for the different colors—so hence the separation. The color separation effect produced by the prism is known as *angular dispersion*, or more specifically, *chromatic* angular dispersion.

The term *dispersion* implies a *separation* of distinguishable components of a wave. In the case of the prism, the components are the various colors that have been spatially separated. An important point here is that the spectral *power* has been dispersed by the prism. We can motivate this idea by considering what it would take to measure the difference in refracted angles between, for example, blue and red light. One would need to use a power detector with a very narrow aperture, as shown in Fig. 12.13. The detector would be positioned at the locations of the blue and red light from the prism, with the narrow aperture allowing essentially one color at a time (or light over a very narrow spectral range) to pass through to the detector. The detector would thus measure the power in what we could call a "spectral packet," or a very narrow slice of the total power spectrum. The smaller the aperture, the narrower the spectral width of the packet, and thus the greater the precision in the measurement.[4] It is important for us to think of wave power as subdivided into spectral packets in this way, as it will figure

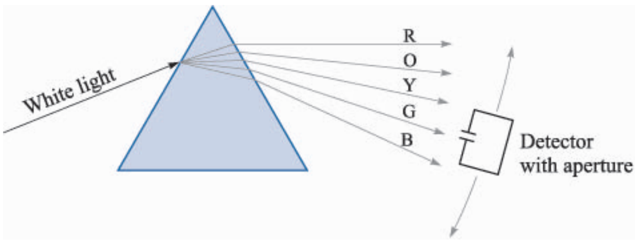

FIGURE 12.13
The angular dispersion of a prism can be measured using a movable device which measures both wavelength and power. The device senses light through a small aperture, thus improving wavelength resolution.

[4] To perform this experiment, one would need to measure the wavelength as well. Thus, the detector would likely be located at the output of a spectrometer or monochrometer, whose input slit performs the function of the bandwidth-limiting aperture.

prominently in our interpretation of the main topic of this section, which is wave dispersion *in time*.

We now consider a lossless nonmagnetic medium in which the refractive index varies with frequency. The propagation constant of a uniform plane wave in this medium will assume the form:

$$\beta(\omega) = k = \omega\sqrt{\mu_0\epsilon(\omega)} = n(\omega)\frac{\omega}{c} \tag{80}$$

If we take $n(\omega)$ to be a monotonically increasing function of frequency (as is usually the case), a plot of ω vs. β would look something like the curve shown in Fig. 12.14. Such a plot is known as an ω-β *diagram* for the medium. Much can be learned about how waves propagate in the material by considering the shape of the ω-β curve.

Suppose we have two waves at two frequencies, ω_a and ω_b, which are co-propagating in the material, and whose amplitudes are equal. The two frequencies are labeled on the curve in Fig. 12.14, along with the frequency mid-way between the two, ω_0. The corresponding propagation constants, β_a, β_b, and β_0 are also labeled. The electric fields of the two waves are linearly polarized in the same direction (along x for example), while both waves propagate in the forward z direction. The waves will thus interfere with each other, producing a resultant wave whose field function can be found simply by adding the **E** fields of the two waves. This addition is done using the complex fields:

$$E_{c,net}(z,t) = E_0\left[e^{-j\beta_a z}e^{j\omega_a t} + e^{-j\beta_b z}e^{j\omega_b t}\right]$$

Note that we must use the full complex forms (with frequency dependence retained) as opposed to the phasor forms, since the waves are at different frequencies. Next, we factor out the term $e^{-j\beta_0 z}e^{j\omega_0 t}$:

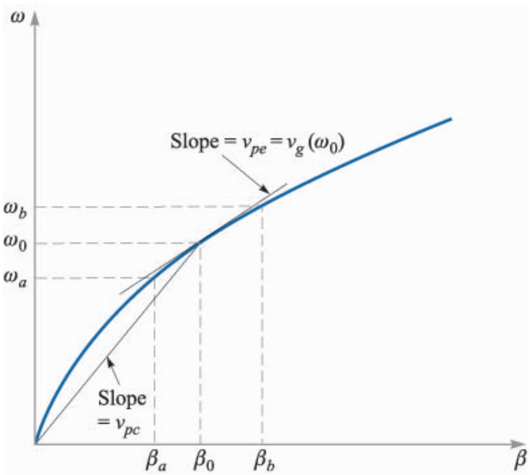

FIGURE 12.14
ω-β diagram for a material in which refractive index increases with frequency. The slope of a line tangent to the curve at ω_0 is the group velocity at that frequency. The slope of a line joining the origin to the point on the curve at ω_0 is the phase velocity at ω_0.

$$E_{c,net}(z,t) = E_0 e^{-j\beta_0 z} e^{j\omega_0 t}\left[e^{j\Delta\beta z}e^{-j\Delta\omega t} + e^{-j\Delta\beta z}e^{j\Delta\omega t}\right] = 2E_0 e^{-j\beta_0 z}e^{j\omega_0 t}\cos(\Delta\omega t - \Delta\beta z) \tag{81}$$

where

$$\Delta\omega = \omega_0 - \omega_a = \omega_b - \omega_0$$

and

$$\Delta\beta = \beta_0 - \beta_a = \beta_b - \beta_0$$

The above expression for $\Delta\beta$ is approximately true as long as $\Delta\omega$ is small. This can be seen from Fig. 12.14, by observing how the shape of the curve affects $\Delta\beta$, given uniform frequency spacings.

The real instantaneous form of (81) is found through

$$E_{net}(z,t) = \text{Re}\{E_{c,net}\} = 2E_0 \cos(\Delta\omega t - \Delta\beta z)\cos(\omega_0 t - \beta_0 z) \tag{82}$$

If $\Delta\omega$ is fairly small compared to ω_0, we recognize (82) as a carrier wave at frequency ω_0 that is sinusoidally modulated at frequency $\Delta\omega$. The two original waves are thus "beating" together to form a slow modulation, as one would hear when the same note is played by two slightly out-of-tune musical instruments. The resultant wave is shown in Fig. 12.15.

Of interest to us are the phase velocities of the carrier wave and the modulation envelope. From (82), we can immediately write these down as:

$$v_{pc} = \frac{\omega_0}{\beta_0} \qquad \text{(carrier velocity)} \tag{83}$$

$$v_{pe} = \frac{\Delta\omega}{\Delta\beta} \qquad \text{(envelope velocity)} \tag{84}$$

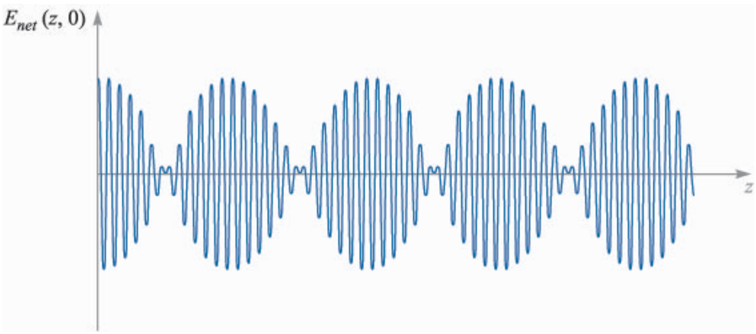

FIGURE 12.15
Plot of the total electric field strength as a function of z (with $t = 0$) of two co-propagating waves having different frequencies, ω_a and ω_b, as per Eq. (82). The rapid oscillations are associated with the carrier frequency, $\omega_0 = (\omega_a + \omega_b)/2$. The slower modulation is associated with the envelope or "beat" frequency, $\Delta\omega = (\omega_b - \omega_a)/2$.

Referring to the ω-β diagram, Fig. 12.14, we recognize the carrier phase velocity as the slope of the straight line that joins the origin to the point on the curve whose coordinates are ω_0 and β_0. We recognize the envelope velocity as a quantity that approximates the slope of the ω-β curve at the location of an operation point specified by (ω_0, β_0). The envelope velocity in this case is thus somewhat less than the carrier velocity. As $\Delta\omega$ becomes vanishingly small, the envelope velocity is identically the slope of the curve at ω_0. We can thus state the following for our example:

$$\boxed{\lim_{\Delta\omega\to 0}\frac{\Delta\omega}{\Delta\beta} = \left.\frac{d\omega}{d\beta}\right|_{\omega_0} = v_g(\omega_0)} \tag{85}$$

The quantity $d\omega/d\beta$ is called the *group velocity* function for the material, $v_g(\omega)$. When evaluated at a specified frequency, ω_0, it represents the velocity of a group of frequencies within a spectral packet of vanishingly small width, centered at frequency ω_0. In stating this, we have extended our two-frequency example to include waves that have a continuous frequency spectrum. To each frequency component (or packet) is associated a group velocity at which the energy in that packet propagates. Since the slope of the ω-β curve changes with frequency, group velocity will obviously be a function of frequency. The *group velocity dispersion* of the medium is, to first order, the rate at which the slope of the ω-β curve changes with frequency. It is this behavior that is of critical practical importance to the propagation of modulated waves within dispersive media, and the extent to which the modulation envelope may degrade with propagation distance.

Example 12.10

Consider a medium in which the refractive index varies linearly with frequency over a certain range:

$$n(\omega) = n_0\frac{\omega}{\omega_0}$$

Determine the group velocity and the phase velocity of a wave at frequency ω_0.

Solution. First, the phase constant will be

$$\beta(\omega) = n(\omega)\frac{\omega}{c} = \frac{n_0\omega^2}{\omega_0 c}$$

Now

$$\frac{d\beta}{d\omega} = \frac{2n_0\omega}{\omega_0 c}$$

so that

$$v_g = \frac{d\omega}{d\beta} = \frac{\omega_0 c}{2n_0\omega}$$

The group velocity at ω_0 is

$$v_g(\omega_0) = \frac{c}{2n_0}$$

The phase velocity at ω_0 will be

$$v_p(\omega_0) = \frac{\omega}{\beta(\omega_0)} = \frac{c}{n_0}$$

To see how a dispersive medium affects a modulated wave, let us consider the propagation of an electromagnetic pulse. Pulses are used in digital signals, where the presence or absence of a pulse in a given time slot corresponds to a digital "one" or "zero." The effect of the dispersive medium on a pulse is to broaden it in time. To see how this happens, we consider the pulse *spectrum*, which is found through the Fourier transform of the pulse in time domain. In particular, suppose the pulse shape in time is Gaussian, and has electric field given at position $z = 0$ by

$$\boxed{E_c(0, t) = E_0 e^{-\frac{1}{2}(t/T)^2} e^{j\omega_0 t}} \tag{86}$$

where E_0 is a constant, ω_0 is the carrier frequency, and where T is the characteristic half-width of the pulse envelope; this is the time at which the pulse *intensity*, or magnitude of the Poynting vector, falls to $1/e$ of its maximum value (note that intensity is proportional to the square of the electric field). The frequency spectrum of the pulse is the Fourier transform of (86), which is

$$\boxed{E(0, \omega) = \frac{E_0 T}{\sqrt{2\pi}} e^{-\frac{1}{2}T^2(\omega-\omega_0)^2}} \tag{87}$$

Note from (87) that the frequency displacement from ω_0 at which the spectral *intensity* (proportional to $|E(0, \omega)|^2$) falls to $1/e$ of its maximum is $\Delta\omega = \omega - \omega_0 = 1/T$.

Fig. 12.16*a* shows the Gaussian intensity spectrum of the pulse, centered at ω_0, where the frequencies corresponding to the $1/e$ spectral intensity positions, ω_a and ω_b, are indicated. Fig. 12.16*b* shows the same three frequencies marked on the ω-β curve for the medium. Three lines are drawn that are tangent to the curve at the three frequency locations. The slopes of the lines indicate the group velocities at ω_a, ω_b, and ω_0, indicated as v_{ga}, v_{gb}, and v_{g0}. We can think of the pulse spreading in time as resulting from the differences in propagation times of the spectral energy packets that make up the pulse spectrum. Since the pulse spectral energy is highest at the center frequency, ω_0, we can use this as a reference point, about which further spreading of the energy will occur. For example, let us consider the difference in arrival times (group delays) between the frequency components, ω_0 and ω_b, after propagating through a distance z of the medium:

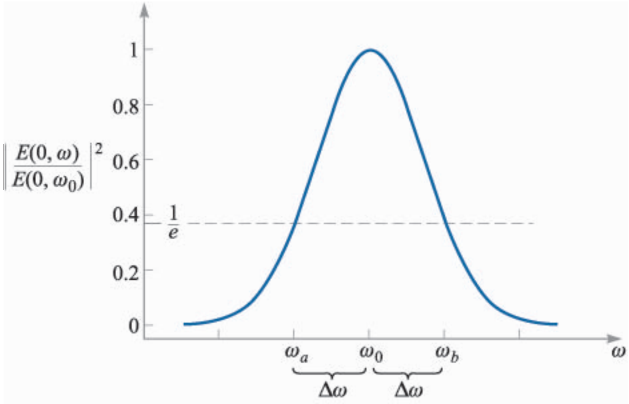

FIGURE 12.16*a*
Normalized power spectrum of a Gaussian pulse, as determined from Eq. (86). The spectrum is centered at carrier frequency ω_0, and has $1/e$ half-width, $\Delta\omega$. Frequencies ω_a and ω_b correspond to the $1/e$ positions on the spectrum.

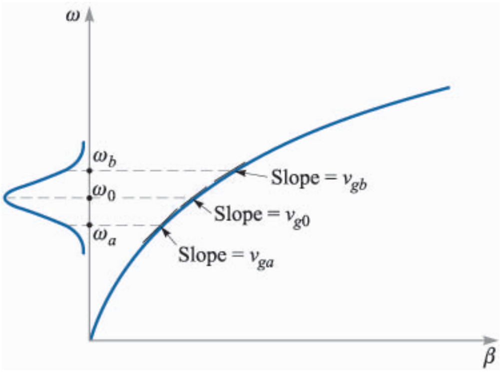

FIGURE 12.16*b*
The spectrum of Fig. 12.16*a* as shown on the ω-β diagram for the medium. The three frequencies specified in Fig. 12.16*a* are associated with three different slopes on the curve, resulting in different group delays for the spectral components.

$$\Delta\tau = z\left(\frac{1}{v_{gb}} - \frac{1}{v_{g0}}\right) = z\left(\left.\frac{d\beta}{d\omega}\right|_{\omega b} - \left.\frac{d\beta}{d\omega}\right|_{\omega 0}\right) \tag{88}$$

The gist of this result is that the medium is acting as what could be called a *temporal prism*. Instead of spreading out the spectral energy packets spatially, it is spreading them out in time. In so doing, a new temporal pulse envelope is constructed whose width is based fundamentally on the spread of propagation delays of the different spectral components. By determining the delay difference between the peak spectral component and the component at the spectral half-width, we construct an expression for the new *temporal* half-width. This assumes, of course, that the initial pulse width is negligible in comparison, but if not, we can account for that also, as will be shown later on.

To evaluate (88), we need more information about the ω-β curve. If we assume that the curve is smooth and has fairly uniform curvature, we can express $\beta(\omega)$ as the first three terms of a Taylor series expansion about the carrier frequency, ω_0:

$$\boxed{\beta(\omega) \doteq \beta(\omega_0) + (\omega - \omega_0)\beta_1 + \frac{1}{2}(\omega - \omega_0)^2 \beta_2} \tag{89}$$

where

$$\beta_0 = \beta(\omega_0)$$

$$\beta_1 = \left.\frac{d\beta}{d\omega}\right|_{\omega 0} \tag{90}$$

and

$$\beta_2 = \left.\frac{d^2\beta}{d\omega^2}\right|_{\omega 0} \tag{91}$$

Note that if the ω-β curve were a straight line, then the first two terms in (89) would precisely describe $\beta(\omega)$. It is the third term in (89), involving β_2, that describes the curvature and ultimately the dispersion.

Noting that β_0, β_1, and β_2 are constants, we take the first derivative of (89) with respect to ω to find

$$\frac{d\beta}{d\omega} = \beta_1 + (\omega - \omega_0)\beta_2 \tag{92}$$

We now substitute (92) into (88) to obtain

$$\boxed{\Delta\tau = [\beta_1 + (\omega_b - \omega_0)\beta_2]z - [\beta_1 + (\omega_0 - \omega_0)\beta_2]z = \Delta\omega\beta_2 z = \frac{\beta_2 z}{T}} \tag{93}$$

where $\Delta\omega = (\omega_b - \omega_0) = 1/T$. β_2, as defined in Eq. (91), is the *dispersion parameter*. Its units are in general [time^2/distance], that is—pulse spread in time per unit spectral bandwidth, per unit distance. In optical fibers, for example, the units most commonly used are picoseconds^2/kilometer (psec^2/km). β_2 can be determined, knowing how β varies with frequency, or it can be measured.

If the initial pulse width is very short compared to $\Delta\tau$, then the broadened pulse width at location z will be simply $\Delta\tau$. If the initial pulse width is comparable to $\Delta\tau$, then the pulse width at z can be found through the convolution of the initial Gaussian pulse envelope of width T with a Gaussian envelope whose width is $\Delta\tau$. Thus in general, the pulse width at location z will be

$$\boxed{T' = \sqrt{T^2 + (\Delta\tau)^2}} \tag{94}$$

Example 12.11

An optical fiber channel is known to have dispersion, $\beta_2 = 20\ \mathrm{ps}^2/\mathrm{km}$. A Gaussian light pulse at the input of the fiber is of initial width $T = 10$ ps. Determine the width of the pulse at the fiber output, if the fiber is 15 km long.

Solution. The pulse spread will be

$$\Delta\tau = \frac{\beta_2 z}{T} = \frac{(20)(15)}{10} = 30\ \mathrm{ps}$$

So the output pulse width is

$$T' = \sqrt{(10)^2 + (30)^2} = 32\ \mathrm{ps}$$

An interesting by-product of pulse broadening through chromatic dispersion is that the broadened pulse is *chirped*. This means that the instantaneous frequency of the pulse varies monotonically (either increases or decreases) with time over the pulse envelope. This again is just a manifestation of the broadening mechanism, in which the spectral components at different frequencies are spread out in time as they propagate at different group velocities. We can quantify the effect by calculating the group delay, τ_g, as a function of frequency, using (92). We obtain:

$$\tau_g = \frac{z}{v_g} = z\frac{d\beta}{d\omega} = (\beta_1 + (\omega - \omega_0)\beta_2)z \tag{95}$$

This equation tells us that the group delay will be a linear function of frequency, and that higher frequencies will arrive at later times, if β_2 is positive. We refer to the chirp as positive if lower frequencies lead the higher frequencies in time (requiring a positive β_2 in (95)); chirp is negative if the higher frequencies lead in time (negative β_2). Fig. 12.17 shows the broadening effect and illustrates the chirping phenomenon.

✔ **D12.6.** For the fiber channel of Example 12.11, a 20 ps pulse is input instead of the 10 ps pulse in the example. Determine the output pulsewidth.

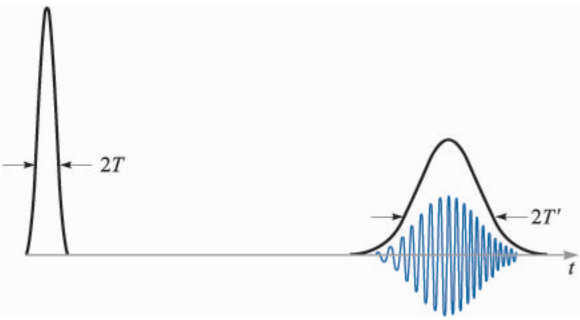

FIGURE 12.17
Gaussian pulse intensities as functions of time (smooth curves) before and after propagation through a dispersive medium, as exemplified by the ω-β diagram of Fig. 12.16*b*. The electric field oscillations are shown under the second trace to demonstrate the chirping effect as the pulse broadens. Note the reduced amplitude of the broadened pulse, which occurs because the pulse energy (the area under the intensity envelope) is constant.

Ans. 25 ps.

As a final point, we note that the pulse bandwidth, $\Delta\omega$, was found to be $1/T$. This is true as long as the Fourier transform of the pulse *envelope* is taken, as was done with (86) to obtain (87). In that case, E_0 was taken to be a constant, and so the only time variation arose from the carrier wave and the Gaussian envelope. Such a pulse, whose frequency spectrum is obtained only from the pulse envelope, is known as *transform-limited*. In general, however, additional frequency bandwidth may be present since E_0 may vary with time for one reason or another (such as phase noise that could be present on the carrier). In these cases, pulse broadening is found from the more general expression

$$\boxed{\Delta\tau = \Delta\omega\beta_2 z} \tag{96}$$

where $\Delta\omega$ is the net spectral bandwidth arising from all sources. Clearly, transform-limited pulses are preferred in order to minimize broadening, since these will have the smallest spectral width for a given pulse width.

SUGGESTED REFERENCES

1. DuBroff, R. E., S. V. Marshall, and G. G. Skitek: ''Electromagnetic Concepts and Applications,'' 4th ed., Prentice Hall, New Jersey, 1996. Chapter 9 of this text develops the concepts presented here, with additional examples and applications.
2. Iskander, M. F.: ''Electromagnetic Fields and Waves,'' Prentice Hall, New Jersey, 1992. The multiple interface treatment in Chapter 5 of this text is particularly good.
3. Harrington, R. F.: ''Time-Harmonic Electromagnetic Fields,'' McGraw-Hill, New York, 1961. This advanced text provides a good overview of general wave reflection concepts in Chapter 2.
4. Marcuse, D.: ''Light Transmission Optics,'' Van Nostrand Reinhold, New York, 1982. This intermediate-level text provides detailed coverage of optical waveguides and pulse propagation in dispersive media.

PROBLEMS

12.1 A uniform plane wave in air, $E_{x1}^+ = E_{x10}^+ \cos(10^{10}t - \beta z)$ V/m, is normally incident on a copper surface at $z = 0$. What percentage of the incident power density is transmitted into the copper?

12.2 The plane $y = 0$ defines the boundary between two different dielectrics. For $y < 0$, $\epsilon'_{R1} = 1$, $\mu_1 = \mu_0$, and $\epsilon''_{R1} = 0$; and for $y > 0$, $\epsilon'_{R2} = 5$, $\mu_2 = \mu_0$, and $\epsilon''_{R2} = 0$. Let $E_{z1}^+ = 150\cos(\omega t - 8y)$ V/m, and find: (a) ω; (b) $\mathbf{H}_1^+$; (c) $\mathbf{H}_1^-$.

12.3 A uniform plane wave in region 1 is normally incident on the planar boundary separating regions 1 and 2. If $\epsilon''_1 = \epsilon''_2 = 0$, while $\epsilon'_{R1} = \mu^3_{R1}$ and $\epsilon'_{R2} = \mu^3_{R2}$, find the ratio $\epsilon'_{R2}/\epsilon'_{R1}$ if 20% of the energy in the incident wave is reflected at the boundary. There are two possible answers.

12.4 The magnetic field intensity in a region where $\epsilon'' = 0$ is given as $\mathbf{H} = 5\cos\omega t \cos\beta z\, \mathbf{a_y}$ A/m, where $\omega = 5$ Grad/s and $\beta = 30$ rad/m. If the amplitude of the associated electric field intensity is 2 kV/m, find: (a) μ and ϵ' for the medium; (b) $\mathbf{E}$.

12.5 The region $z < 0$ is characterized by $\epsilon'_R = \mu_R = 1$ and $\epsilon''_R = 0$. The total $\mathbf{E}$ field here is given as the sum of two uniform plane waves, $\mathbf{E_s} = 150\, e^{-j10z}\, \mathbf{a_x} + (50\angle 20^\circ)\, e^{j10z}\, \mathbf{a_x}$ V/m. (a) What is the operating frequency? (b) Specify the intrinsic impedance of the region $z > 0$ that would provide the appropriate reflected wave. (c) At what value of z, $-10\,\text{cm} < z < 0$, is the total electric field intensity a maximum amplitude?

12.6 Region 1, $z < 0$, and region 2, $z > 0$, are described by the following parameters: $\epsilon'_1 = 100$ pF/m, $\mu_1 = 25$ μH/m, $\epsilon''_1 = 0$, $\epsilon'_2 = 200$ pF/m, $\mu_2 = 50$ μH/m, and $\epsilon''_2/\epsilon'_2 = 0.5$. If $\mathbf{E}^+_1 = 600\, e^{-\alpha_1 z} \cos(5 \times 10^{10} t - \beta_1 z)$ $\mathbf{a_x}$ V/m, find: (a) α_1; (b) β_1; (c) $\mathbf{E}^+_{s1}$; (d) $\mathbf{E}^-_{s1}$; (e) $\mathbf{E}^+_{s2}$.

12.7 The semi-infinite regions, $z < 0$ and $z > 1$ m, are free space. For $0 < z < 1$ m, $\epsilon'_R = 4$, $\mu_R = 1$, and $\epsilon''_R = 0$. A uniform plane wave with $\omega = 4 \times 10^8$ rad/s is traveling in the $\mathbf{a_z}$ direction toward the interface at $z = 0$. (a) Find the standing wave ratio in each of the three regions. (b) Find the location of the maximum $|\mathbf{E}|$ for $z < 0$ that is nearest to $z = 0$.

12.8 A wave starts at point a, propagates 100 m through a lossy dielectric for which $\alpha = 0.5$ Np/m, reflects at normal incidence at a boundary at which $\Gamma = 0.3 + j0.4$, and then returns to point a. Calculate the ratio of the final power to the incident power after this round trip.

12.9 Region 1, $z < 0$, and region 2, $z > 0$, are both perfect dielectrics ($\mu = \mu_0$, $\epsilon'' = 0$). A uniform plane wave traveling in the $\mathbf{a_z}$ direction has a radian frequency of 3×10^{10} rad/s. Its wavelengths in the two regions are $\lambda_1 = 5$ cm and $\lambda_2 = 3$ cm. What percentage of the energy incident on the boundary is: (a) reflected; (b) transmitted? (c) What is the standing wave ratio in region 1?

12.10 In Fig. 12.1, let region 2 be free space, while $\mu_{R1} = 1$, $\epsilon''_{R1} = 0$, and ϵ'_{R1} is unknown. Find ϵ'_{R1} if: (a) the amplitude of $\mathbf{E}^-_1$ is one-half that of $\mathbf{E}^+_1$; (b) $\mathcal{P}^-_{1,av}$ is one-half of $\mathcal{P}^+_{1,av}$; (c) $|\mathbf{E}_1|_{min}$ is one-half of $|\mathbf{E}_1|_{max}$.

12.11 A 150 MHz uniform plane wave is normally incident from air onto a material whose intrinsic impedance is unknown. Measurements yield a standing wave ratio of 3 and the appearance of an electric field minimum at 0.3 wavelengths in front of the interface. Determine the impedance of the unknown material.

12.12 A 50 MHz uniform plane wave is normally incident from air onto the surface of a calm ocean. For seawater, $\sigma = 4$ S/m, and $\epsilon'_R = 78$. (a) Determine the fractions of the incident power that are reflected and

transmitted. (b) Qualitatively, how will these answers change (if at all) as the frequency is increased?

12.13 A right-circularly polarized plane wave is normally incident from air onto a semi-infinite slab of plexiglas ($\epsilon'_R = 3.45$, $\epsilon''_R = 0$). Calculate the fractions of the incident power that are reflected and transmitted. Also, describe the polarizations of the reflected and transmitted waves.

12.14 A left-circularly polarized plane wave is normally incident onto the surface of a perfect conductor. (a) Construct the superposition of the incident and reflected waves in phasor form. (b) Determine the real instantaneous form of the result of part a. (c) Describe the wave that is formed.

12.15 Consider these regions in which $\epsilon'' = 0$: region 1, $z < 0$, $\mu_1 = 4\ \mu\text{H/m}$ and $\epsilon'_1 = 10$ pF/m; region 2, $0 < z < 6$ cm, $\mu_2 = 2\ \mu\text{H/m}$, $\epsilon'_2 = 25$ pF/m; region 3, $z > 6$ cm, $\mu_3 = \mu_1$ and $\epsilon'_3 = \epsilon'_1$. (a) What is the lowest frequency at which a uniform plane wave incident from region 1 onto the boundary at $z = 0$ will have no reflection? (b) If $f = 50$ MHz, what will the standing wave ratio be in region 1?

12.16 A uniform plane wave in air is normally incident onto a lossless dielectric plate of thickness $\lambda/8$, and of intrinsic impedance $\eta = 260\ \Omega$. Determine the standing wave ratio in front of the plate. Also find the fraction of the incident power that is transmitted to the other side of the plate.

12.17 Repeat Problem 12.16 for the cases in which the frequency is (a) doubled, and (b) quadrupled. Assume that the slab impedance is independent of frequency.

12.18 In Fig. 12.6, let $\eta_1 = \eta_3 = 377\,\Omega$, and $\eta_2 = 0.4\eta_1$. A uniform plane wave is normally incident from the left, as shown. Plot a curve of the standing wave ratio, s in the region to the left: (a) as a function of l if $f = 2.5$ GHz; (b) as a function of frequency if $l = 2$ cm.

12.19 You are given four slabs of lossless dielectric, all with the same intrinsic impedance, η, known to be different from that of free space. The thickness of each slab is $\lambda/4$, where λ is the wavelength as measured in the slab material. The slabs are to be positioned parallel to one another, and the combination lies in the path of a uniform plane wave, normally incident. The slabs are to be arranged such that the air spaces between them are either zero, one-quarter wavelength, or one-half wavelength in thickness. Specify an arrangement of slabs and air spaces such that (a) the wave is totally transmitted through the stack, and (b) the stack presents the highest reflectivity to the incident wave. Several answers may exist.

12.20 The 50 MHz plane wave of Problem 12.12 is incident onto the ocean surface at an angle to the normal of 60°. Determine the fractions of the incident power that are reflected and transmitted for (a) s polarization, and (b) p polarization.

12.21 A right-circularly polarized plane wave in air is incident at Brewster's angle onto a semi-infinite slab of plexiglas ($\epsilon'_R = 3.45$, $\epsilon''_R = 0$). (a) Determine the fractions of the incident power that are reflected and

transmitted. (b) Describe the polarizations of the reflected and transmitted waves.

12.22 A dielectric waveguide is shown in Fig. 12.18 with refractive indices as labeled. Incident light enters the guide at angle ϕ from the front surface normal as shown. Once inside, the light totally reflects at the upper $n_1 - n_2$ interface, where $n_1 > n_2$. All subsequent reflections from the upper and lower boundaries will be total as well, and so the light is confined to the guide. Express, in terms of n_1 and n_2, the maximum value of ϕ such that total confinement will occur, with $n_0 = 1$. The quantity $\sin\phi$ is known as the *numerical aperture* of the guide.

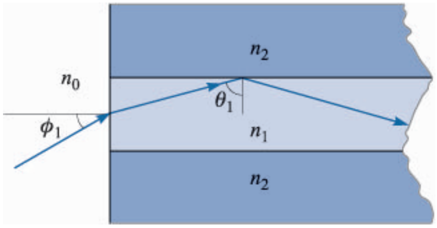

FIGURE 12.18
See Problems 12.22 and 12.23.

12.23 Suppose that ϕ in Fig. 12.18 is Brewster's angle, and that θ_1 is the critical angle. Find n_0 in terms of n_1 and n_2.

12.24 A *Brewster prism* is designed to pass p-polarized light without any reflective loss. The prism of Fig. 12.19 is made of glass ($n = 1.45$), and is in air. Considering the light path shown, determine the apex angle, α.

12.25 In the Brewster prism of Fig. 12.19, determine for s-polarized light the fraction of the incident power that is transmitted through the prism.

12.26 Show how a single block of glass can be used to turn a p-polarized beam of light through 180°, with the light suffering (in principle) zero reflective loss. The light is incident from air, and the returning beam (also in air) may be displaced sideways from the incident beam. Specify all pertinent angles and use $n = 1.45$ for glass. More than one design is possible here.

12.27 Using Eq. (59) in Chapter 11 as a starting point, determine the ratio of the group and phase velocities of an electromagnetic wave in a good conductor. Assume conductivity does not vary with frequency.

12.28 Over a certain frequency range, the refractive index of a certain material varies approximately linearly with frequency:

$$n(\omega) \doteq n_a + n_b(\omega - \omega_a)$$

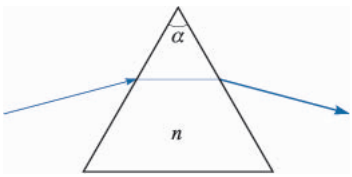

FIGURE 12.19
See Problems 12.24 and 12.25.

where n_a, n_b, and ω_a are constants. Using $\beta = n\omega/c$: (a) determine the group velocity as a function (or perhaps not a function) of frequency; (b) determine the group dispersion parameter, β_2; (c) discuss the implications of these results, if any, on pulse broadening.

12.29 A $T = 5$ ps transform-limited pulse propagates in a dispersive channel for which $\beta_2 = 10\,\text{ps}^2/\text{km}$. Over what distance will the pulse spread to twice its initial width?

12.30 A $T = 20$ ps transform-limited pulse propagates through 10 km of a dispersive channel for which $\beta_2 = 12\,\text{ps}^2/\text{km}$. The pulse then propagates through a second 10 km channel for which $\beta_2 = -12\,\text{ps}^2/\text{km}$. Describe the pulse at the output of the second channel and give a physical explanation for what happened.